

Optimization Problems

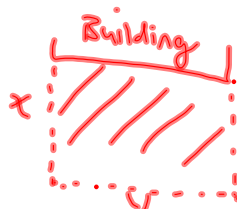
In optimization problems, in general, we are looking for the largest and/or smallest that a function can be. We saw how to one kind of optimization problem in the Absolute Extrema section where we found the largest and smallest value that a function would take on an interval.

In this section we are going to look at another type of optimization problem. Here we will be looking for the largest or smallest values of a function subject to some kind of constraint. It's usually easiest to see how these work with some examples.

Example 1:

We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

- Draw a sketch
- Determine the constraint
- ~~Determine a function~~ in terms of a single variable
- * Determine the absolute maximum value of this function
- Test solution with second derivative



$A = xy$
 $A = x(500 - 2x)$
 ~~$A = 500x - 2x^2$~~
 $2x + y = 500$
 $y = 500 - 2x$
 $A = -2(x^2 - 250x + 15625) + 31250$
 $A = -2(x - 125)^2 + 31250$
 $V(125, 31250)$
 (x, A)
125m by 250m

Calculus Method:

$$A = 500x - 2x^2$$

$$A' = 500 - 4x$$

$$0 = 500 - 4x$$

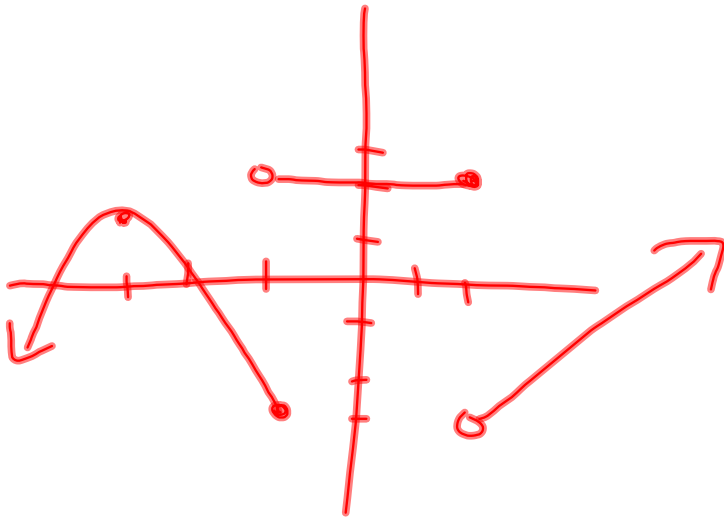
$$4x = 500$$

$$x = 125 \rightarrow y = 500 - 2x$$

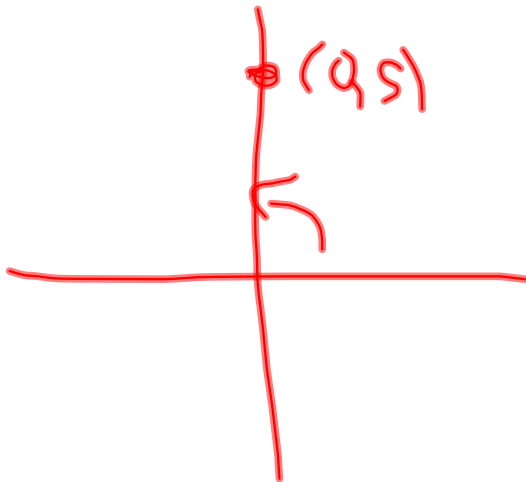
$$y = 500 - 2(125)$$

$$y = 250$$

125m by 250m



(s, i)
 $(0, s)$

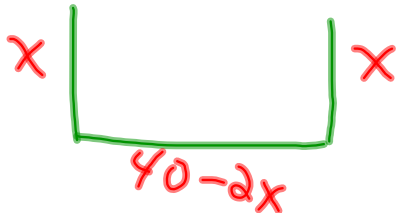


Example 2:

A piece of tin 40 cm wide is to be folded up as shown.

How deep will this gutter be if it is to have maximum carrying capacity?

Maximize area of face



$$A = x(40 - 2x)$$

$$A = 40x - 2x^2$$

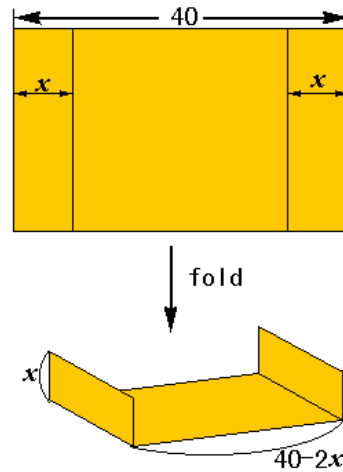
$$A' = 40 - 4x$$

$$0 = 40 - 4x$$

$$4x = 40$$

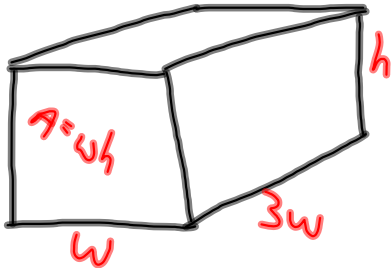
$$x = 10$$

10 cm deep



Example 3:

We are going to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft² and the material used to build the sides cost \$6/ft². If the box must have a volume of 50ft³ determine the dimensions that will minimize the cost to build the box.



$$V = 3w^2h$$

$$50 = 3w^2h$$

$$h = \frac{50}{3w^2}$$

$$\text{Cost} = \overset{\text{Front/Back}}{(6wh) \times 2} + \overset{\text{Left/Right}}{6(3wh)(2)} + \overset{\text{Top/Bottom}}{10(3w^2)(2)}$$

$$C = 48wh + 60w^2$$

$$C = 48w \left(\frac{50}{3w^2} \right) + 60w^2$$

$$C = \overset{800w^{-1}}{\frac{800}{w}} + 60w^2$$

$$C' = -800w^{-2} + 120w$$

$$0 = -\frac{800}{w^2} + 120w$$

$$0 = -800 + 120w^3$$

$$\frac{-120w^3}{-120} = \frac{-800}{-120}$$

$$\sqrt[3]{w^3} = \sqrt[3]{6.\bar{6}}$$

$$w = \underline{1.88 \text{ ft.}}$$

$$\text{length} = 1.88 \times 3$$

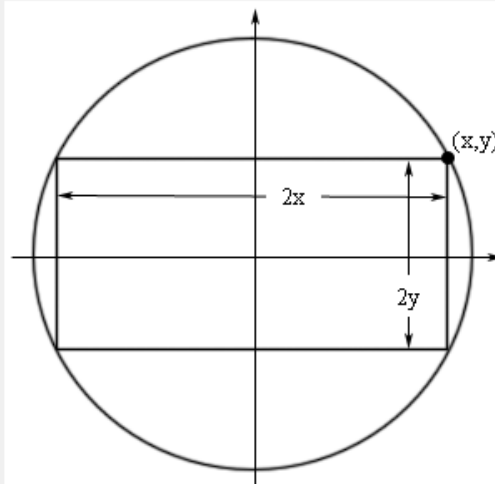
$$= 5.65 \text{ feet}$$

$$\text{height} = \frac{50}{3(1.88)^2} = 4.72 \text{ feet}$$

Example 4:

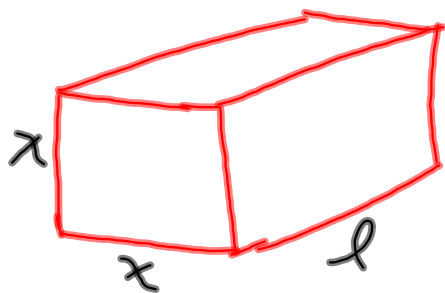
Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

This problem is best described with a sketch. Here is what we're looking for.



Warm Up

The U.S. postal service will accept a box for shipment only if the sum of the length and the girth (*distance around box*) does not exceed 84 inches. Find the volume of the largest box having a square end that could be shipped.



$$V = x^2 l$$

$$V = x^2 (84 - 4x)$$

$$V = 84x^2 - 4x^3$$

$$4x + l = 84$$

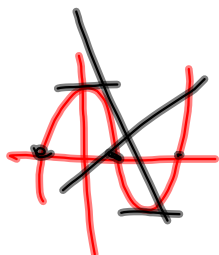
$$l = 84 - 4x$$

$$V' = 168x - 12x^2$$

$$0 = -12x(14 + x)$$

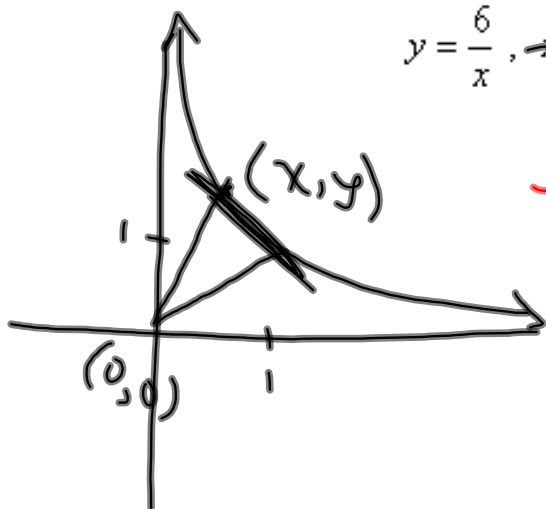
$$x = 0 \text{ or } x = 14 \text{ Maximum}$$

$$V = 84(14)^2 - 4(14)^3 = 5488 \text{ in.}^3$$



Determine the coordinates of the point closest to the origin on the graph of

$$y = \frac{6}{x}, \quad x > 0$$



$$-6x^{-2} = -1$$

$$x^{-2} = \frac{1}{6}$$

$$x^2 = 6$$