Midterm $\frac{}{79}$

Advanced Mathematics with Calculus 120

April 2006

1. Use the definition of a derivative (First Principle) to determine the derivative of $f(x) = \sqrt{2x-1}$. [6]

(No Other Method Accepted !)

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{2x+2h-1} - \sqrt{2x-1}}{h} \left(\frac{\sqrt{2x+2h-1} + \sqrt{2x-1}}{\sqrt{2x+2h-1} + \sqrt{2x-1}} \right)$$

$$\lim_{h \to 0} \frac{(2x+2h-1) - (2x-1)}{h\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$\lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2x+2h-1} + \sqrt{2x-1}\right)}$$

$$= \frac{2}{\sqrt{2x-1} + \sqrt{2x-1}}$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{2x-1}}$$

$$\boxed{= \frac{1}{\sqrt{2x-1}}}$$

2. Evaluate each of the following limits, if they exist: [16]

(a) $\lim\limits_{x \to 0} \frac{\sin^3 8x}{27x^3}$

$$\lim_{x \to 0} \left(\frac{\sin 8x}{8x} \right)^3 \frac{512}{27}$$

$$= \boxed{\frac{512}{27}}$$

(b) $\lim\limits_{w \to 5} \frac{\frac{1}{w} - \frac{1}{5}}{25 - w^2}$

$$\lim_{w \to 5} \frac{\cancel{5-w}}{5w} \times \frac{1}{\cancel{(5-w)}(5+w)}$$

$$= \frac{1}{25(10)}$$

$$= \boxed{\frac{1}{250}}$$

(c) $\lim\limits_{x \to -2} \frac{x^3 + 8}{x + 2}$

$$\lim_{x \to -2} \frac{\cancel{(x+2)}(x^2 - 2x + 4)}{\cancel{x+2}}$$

$$= (-2)^2 - 2(-2) + 4$$

$$= 4 + 4 + 4$$

$$\boxed{= 12}$$

(d) $\lim\limits_{x \to \infty} \frac{25x^4 - 1}{1 - x^4}$

$$\lim_{x \to \infty} \frac{\frac{25x^4}{x^4} - \frac{1}{x^4}}{\frac{1}{x^4} - \frac{x^4}{x^4}}$$

$$\lim_{x \to \infty} \frac{25 - 0}{0 - 1}$$

$$= \boxed{-25}$$

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3. Find the first derivative of each of the following functions. Do not simplify your answers. [12]

(a) $f(x)=(\tan x)\sqrt{4+9x^2}$

$f'(x)=\sec^2 x\sqrt{4+9x^2}+\tan x\left[\frac{1}{2}(4+9x^2)^{-1/2}(18x)\right]$

(b) $h(x)=4\sqrt{x}-\frac{2}{\sqrt[5]{x}}+x^{-2}-e^5$

$h(x)=4x^{1/2}-2x^{-1/5}+x^{-2}-e^5$

$h'(x)=2x^{-1/2}+\frac{2}{5}x^{-6/5}-2x^{-3}$

(c) $y=\frac{\sin^3 3x}{2+\sec x^3}$

$$y'=\frac{3(\sin 3x)^2(\cos 3x(3))(2+\sec x^3)-\sin^3 3x\,(\sec x^3\tan x^3(3x^2))}{(2+\sec x^3)^2}$$

4. (a) Given $f(x)=\sqrt{1-2x}$, evaluate $f'''(-4)$. [5]

$f'(x)=\frac{1}{2}(1-2x)^{-1/2}(-2)$

$f'(x)=-(1-2x)^{-1/2}$

$f''(x)=\frac{1}{2}(1-2x)^{-3/2}(-2)$

$f''(x)=-(1-2x)^{-3/2}$

$f'''(x)=\frac{3}{2}(1-2x)^{-5/2}(-2)$

$f'''(x)=-3(1-2x)^{-\frac{5}{2}}$

$f'''(-4)=-3(1-2(-4))^{-5/2}$

$=-3(9)^{-5/2}$

$=\frac{-3}{9^{5/2}}$

$=\frac{-3}{(\sqrt{9})^5}=\boxed{\frac{-1}{81}}$

≈ -0.0123

(b) Given the curve $x^3+y^3=6xy$, find the equation of the tangent line at (3,3). [5]

$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$

$(3y^2-6x)\frac{dy}{dx}=6y-3x^2$

$\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}$

at (3,3) $m=\frac{18-3(9)}{27-18}$

$m=\frac{-9}{9}=-1$

$y-3=-1(x-3)$

$y-3=-x+3$

$\boxed{x+y-6=0}$

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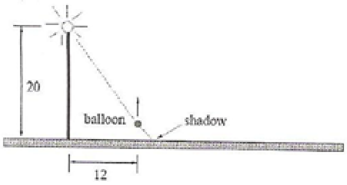
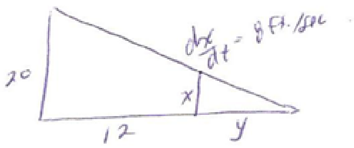
5. A small helium balloon is rising at the rate of 8 ft./sec, a horizontal distance of 12 feet from a 20 ft. lamppost. Determine the rate at which the shadow of the balloon is moving along the ground when it is 5 feet above the ground. [8]





$\frac{20}{x} = \frac{12+y}{y}$

$20y = 12x + xy$

$20\frac{dy}{dt} = 12\frac{dx}{dt} + \frac{dx}{dt}y + x\frac{dy}{dt}$

$20\frac{dy}{dt} = 12(8) + 8(4) + 5\frac{dy}{dt}$

$15\frac{dy}{dt} = 128$

$\boxed{\frac{dy}{dt} = 8.53 \text{ feet/sec.}}$

when $x = 5$ . . .

$\frac{20}{5} = \frac{12+y}{y}$

$20y = 60 + 5y$

$15y = 60$

$y = 4$

6. (a) Find the equation of the normal drawn to $y = \left(\frac{x+3}{3x-6}\right)^3$ at $x = 3$. [6]

$y' = 3\left(\frac{x+3}{3x-6}\right)^2\left(\frac{1(3x-6) - (x+3)(3)}{(3x-6)^2}\right)$

$y = \left(\frac{3+3}{3}\right)^3$

$y = 8$

$(3, 8)$

at $x = 3$ . . $m = 3\left(\frac{6}{3}\right)^2\left(\frac{3 - 18}{3^2}\right)$

$m = 3(2)^2\left(\frac{-15}{9}\right)$

$m = -20$

∴ Slope of normal $= \frac{1}{20}$

$y - 8 = \frac{1}{20}(x - 3)$

$\cancel{4y - 16 = x - 3}$

$\cancel{\boxed{x - 4y + 13 = 0}}$

$20y - 160 = x - 3$

$\boxed{x - 20y + 157 = 0}$

(b) Given $f(x) = \begin{cases} 3 & if \quad x \le -2 \\ x^2 & if \quad -2 < x \le 1 \\ 3-k & if \quad x \ge 1 \end{cases}$. Find each of the following:

(i) $\lim_{x \to -2^-} f(x) =$ 3 [1]

(ii) $\lim_{x \to -2^+} f(x) =$ 4 [1]

(iii) $f(-2) =$ 3 [1]

(iv) State the value of k for which $f(x)$ is continuous at $x = 1$. [2]

$3 - K = 1$

$\boxed{[illegible] = K}$

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7. A billiard ball is hit and travels in a straight line according to the formula $s(t) = 40t^2 + 60t$, where s is the distance traveled in centimeters, and t is the time in seconds. How fast is the ball traveling when it hits the cushion 40 cm from it's initial position? [6]

$40t^2 + 60t = 40$

$40t^2 + 60t - 40 = 0$

$2t^2 + 3t - 2 = 0$

$2t^2 + \cancel{4t}\ t - 2 = 0$

$2t(t+2) - 1(t+2) = 0$

$(2t-1)(t+2) = 0$

$t = \frac{1}{2}$ or -2

Need velocity at $t = \frac{1}{2}$

$s'(t) = 80t + 60$

$s'(\frac{1}{2}) = 80(\frac{1}{2}) + 60$

$\boxed{= 100\text{ cm/sec}}$

8. Given the function $f(x) = x^4 - 6x^2 + 3\ldots$ [10]

(i) Determine the intervals of increase and decrease for the function.
(ii) Determine all local maxima or minima points.
(iii) Determine the absolute maximum of $f(x)$ on the closed interval [-1,2].

$F'(x) = 4x^3 - 12x$

$4x^3 - 12x = 0$

$4x(x^2 - 3) = 0$

$x = 0, \pm\sqrt{3}$

$9 - 6(3) + 3$

	$4x$	$x-\sqrt{3}$	$x+\sqrt{3}$	F'	F
$(-\infty, -\sqrt{3})$	−	−	−	−	Dec
$(-\sqrt{3}, 0)$	−	−	+	+	Inc
$(0, \sqrt{3})$	+	−	+	−	Dec
$(\sqrt{3}, \infty)$	+	+	+	+	Inc

(iii) Absolute Max.

$f(-1) = -2$

$F(0) = 3$

$F(\sqrt{3}) = -6$

$F(2) = -5$

$\boxed{\text{Absolute Max} = 3}$

(i) Increase: $(-\sqrt{3}, 0)$ & $(\sqrt{3}, \infty)$

Decrease: $(-\infty, -\sqrt{3})$ & $(0, \sqrt{3})$

(ii) Local Max: $(0, 3)$

Local Min: $(-\sqrt{3}, -6)$ & $(\sqrt{3}, -6)$

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