Warm Up

Quiz: Derivative Rules

1. $f(x) = (x^3 - 5\sqrt{x})^7 (6x^{-3} + 2x)\sqrt{5\sqrt{x} - \frac{1}{x^2} + e^3}$

$f'(x) = 7(x^3 - 5\sqrt{x})^6 (3x^2 - \frac{5}{2}x^{-1/2})(6x^{-3} + 2x)\sqrt{5\sqrt{x} - \frac{1}{x^2} + e^3}$

$+ (-18x^{-4} + 2)(x^3 - 5\sqrt{x})^7 \sqrt{5\sqrt{x} - \frac{1}{x^2} + e^3}$

$+ \frac{1}{2}(5\sqrt{x} - x^{-2} + e^3)^{-1/2} (\frac{5}{2}x^{-1/2} + 2x^{-3})(x^3 - 5\sqrt{x})^7 (6x^{-3} + 2x)$

2. $g(x) = \sqrt[3]{7x^4 - \sqrt{2x + \sqrt[3]{5x} + 4x^3}}$

$$5x^{1/3}$$

$$(5x)^{1/3}$$

$$g(x) = \left[7x^4 - \left[2x + (5x)^{1/3} + 4x^3\right]^{1/2}\right]^{1/3}$$

or $\sqrt[3]{5}\,x^{1/3} = \frac{\sqrt[3]{5}}{3}x^{-2/3}$

$$g'(x) = \frac{1}{3}\left[7x^4 - \left(2x + (5x)^{1/3} + 4x^3\right)^{1/2}\right]^{-2/3}\left[28x^3 - \frac{1}{2}\left(2x + (5x)^{1/3} + 4x^3\right]^{-1/2} \cdot \left(2 + \frac{1}{3}(5x)^{-2/3}(5) + 12x^2\right)\right]$$

3. $h(x) = \frac{(3x-5)^7 + 8\sqrt{x}\left(x^4+7x\right)^5}{x^3\sqrt{2x-5}}$

$$h'(x) = \frac{\left[7(3x-5)^6(3) + 4x^{-1/2}(x^4+7x)^5 + 8\sqrt{x}\left[5(x^4+7x)^4(4x^3+7)\right]\right]\left[x^3\sqrt{2x-5}\right] - \left[(3x-5)^7 + 8\sqrt{x}(x^4+7x)^5\right] \cdot \left[3x^2\sqrt{2x-5} + x^3\left[\frac{1}{2}(2x-5)^{-1/2}(2)\right]\right]}{\left(x^3\sqrt{2x-5}\right)^2}$$

Differentiate:

$$\left[\cos(7x^3)\right]^3 \qquad (\csc x^8)^4$$

$$y = \frac{\cos^3(7x^3) - \tan(\sin\sqrt{x})}{\sec(3x^2-5) + \sqrt{\cot(7x+2) - \csc^4 x^8}}$$

$$y' = \frac{\left[3(\cos(7x^3))^2\left(-\sin(7x^3)\right)(21x^2) - \sec^2(\sin\sqrt{x})(\cos\sqrt{x})\left(\tfrac{1}{2}x^{-1/2}\right)\right]\cdot\left[\sec(3x^2-5) + (\cot(7x+2) - \csc^4 x^8)^{1/2}\right] - \left(\cos^3 7x^3 - \tan(\sin\sqrt{x})\right)\cdot\left[\sec(3x^2-5)\tan(3x^2-5)(6x) + \tfrac{1}{2}\left[\cot(7x+2) - \csc^4 x^8\right]^{-1/2}\left[-\csc^2(7x+2)(7) - 4(\csc x^8)^3\left(-\csc x^8 \cot x^8 (8x^7)\right)\right]\right]}{\left[\sec(3x^2-5) + \sqrt{\cot(7x+2) - \csc^4 x^8}\right]^2}$$

Implicit Differentiation

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x.
- Examples
  - $x^2 + y^2 = 25$
  - $x^3 + y^3 = 6xy$

- The first equation could easily be rearranged for $y = \ldots$

$$y = \pm\sqrt{25 - x^2}$$

Actually gives two functions

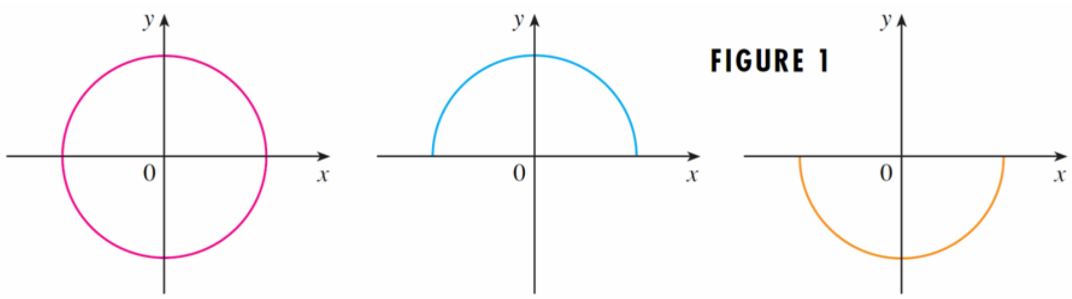

FIGURE 1

(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

- The other sample equation

$$x^3 + y^3 = 6xy$$

  - <u>can</u> be solved for y but
  - the results are very <u>complicated</u>.

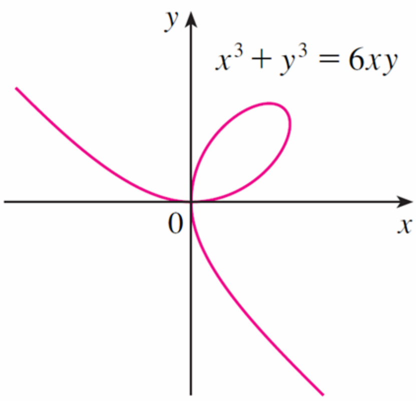

FIGURE 2 The folium of Descartes

$y^3 - 6xy = -x^3$

$y(y^2 - 6x) = -x^3$

Implicit Differentiation

- There is a way called *implicit differentiation* to find *dy/dx* without solving for *y* :
  - First differentiate both sides of the equation with respect to *x* ;
  - Then solve the resulting equation for *y′* .
- We will always assume that the given equation does indeed define *y* as a differentiable function of *x* .

Example

- For the circle $x^2 + y^2 = 25$, find
  - a) *dy/dx*
  - b) an equation of the tangent at the point (3, 4) .

d

$x^2 = 2x$

$(x)^2 = 2(x)^1(1)$

$(y)^3 = 3(y)^2 \frac{dy}{dx}$

$\frac{dy}{dx}$

Solution:

Start by differentiating both sides of the equation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus… $2x + 2y\frac{dy}{dx} = 0$ $\Rightarrow \frac{\cancel{2y}\frac{dy}{dx}}{\cancel{2y}} = \frac{-\cancel{2}x}{\cancel{2}y}$

$\frac{dy}{dx} = \frac{-x}{y}$

Solving for $\frac{dy}{dx}$ … $\frac{dy}{dx} = -\frac{x}{y}$

Therefore at the point (3,4) the equation of the tangent would be…

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Example

- For the folium of Descartes $x^3 + y^3 = 6xy$,
  - Find y'
  - Find the tangent to the curve at the point (3, 3)
  - At what points on the curve is the tangent line horizontal?

$$x^3 + y^3 = 6xy \qquad 6\left(1y + x\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{\frac{dy}{dx}(3y^2-6x)}{(3y^2-6x)} = \frac{6y-3x^2}{3y^2-6x}$$

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}$$

Example:

Given $x^2 - 3x^3y^2 + y^2 = 5xy^3 - 4$

Find $\dfrac{dy}{dx}$

$$2x - \left[9x^2y^2 + 3x^3(2y)\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 5y^3 + 5x(3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(-6x^3y + 2y - 15xy^2\right) = 5y^3 - 2x + 9x^2y^2$$

$$\frac{dy}{dx} = \frac{5y^3 - 2x + 9x^2y^2}{-6x^3y + 2y - 15xy^2}$$