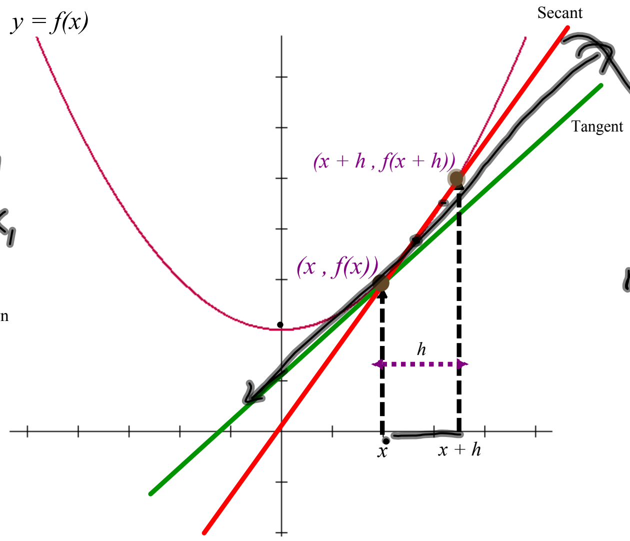
Tangents, Velocities, and Rates of Change

Slope of a tangent to a curve:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

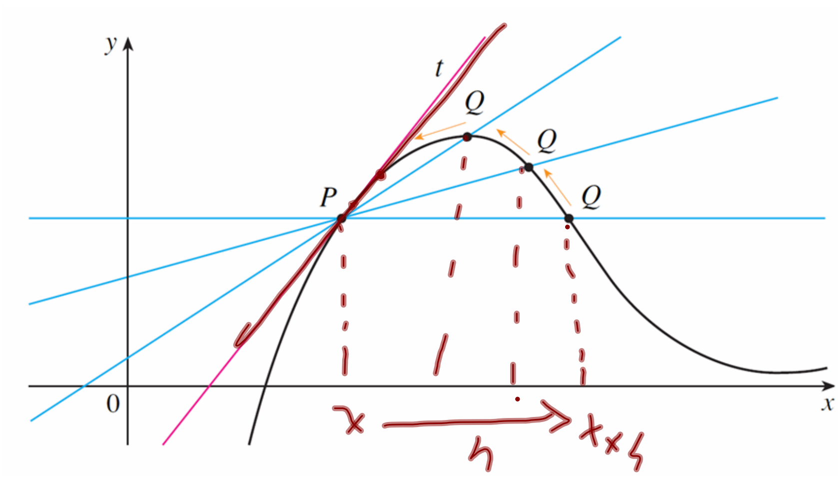


$$m = \frac{f(x+h) - f(x)}{x+h-x}$$

$$m = \frac{f(x+h) - f(x)}{h}$$

How will we find the slope of a tangent drawn to a curve at a point x?

How will the slope of this secant become a better approximation for the slope of the tangent line?



Use your knowledge of limits to determine an expression for that would represent the slope of the tangent line drawn at the point x.

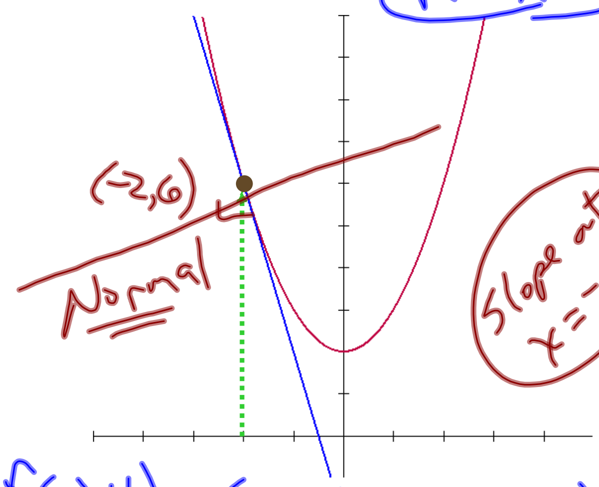
Slope of a Tangent:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example:

Determine the equation of the tangent line drawn to the curve $y = x^2 + 2$ at the point $x = -2$.

$f(x) = x^2+2$



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}$$

Slope at $x = -2$

$$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$f(-2+h) = (-2+h)^2 + 2$$
$$= 4 - 4h + h^2 + 2$$
$$= h^2 - 4h + 6$$

$$f(-2) = (-2)^2 + 2$$
$$= 6$$

$$\lim_{h \to 0} \frac{(h^2 - 4h + 6) - (6)}{h}$$

$$\lim_{h \to 0} \frac{h^2 - 4h}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(h-4)}{\cancel{h}}$$

$$m = -4$$

Equation of Tangent:

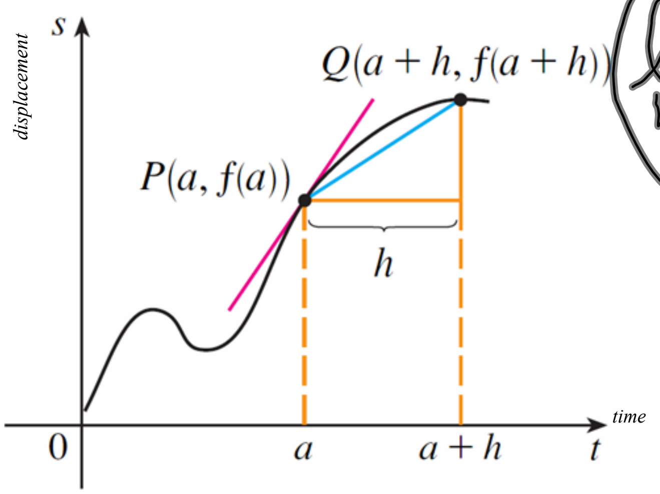
$$y - y_1 = m(x - x_1)$$
$$y - 6 = -4(x+2)$$
$$y - 6 = -4x - 8$$
$$\boxed{y = -4x - 2}$$

What about other applications of limits?

slope . . . .

Rate of change

Velocity...



$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{\Delta s}{\Delta t} = \text{velocity}$$

$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$

$$= \text{average velocity}$$

FIGURE 6

What if we were looking for instantaneous velocity?

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- This means that the
  - velocity at time $t = a$ is equal to the
  - slope of the tangent line at P .

Warm Up

Given that $f(x) = -2x^2 + 5x - \sqrt{x}$, determine the value of…

(1) $f(4)$ (2) $f(\$)$ (3) $f(9+h)$

$f(4) = -2(4)^2 + 5(4) - \sqrt{4}$

$= -32 + 20 - 2$

$= -14$

$f(\$) = -2(\$)^2 + 5(\$) - \sqrt{\$}$

$f(9+h) = -2(9+h)^2 + 5(9+h) - \sqrt{9+h}$

$= -2(81 + 18h + h^2) + 45 + 5h - \sqrt{9+h}$

$= -162 - 36h - 2h^2 + 45 + 5h - \sqrt{9+h}$

$= -2h^2 - 31h - 117 - \sqrt{9+h}$

$\sqrt{9+h} \neq \sqrt{9} + \sqrt{h}$

Develop the definition of a derivative

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change).

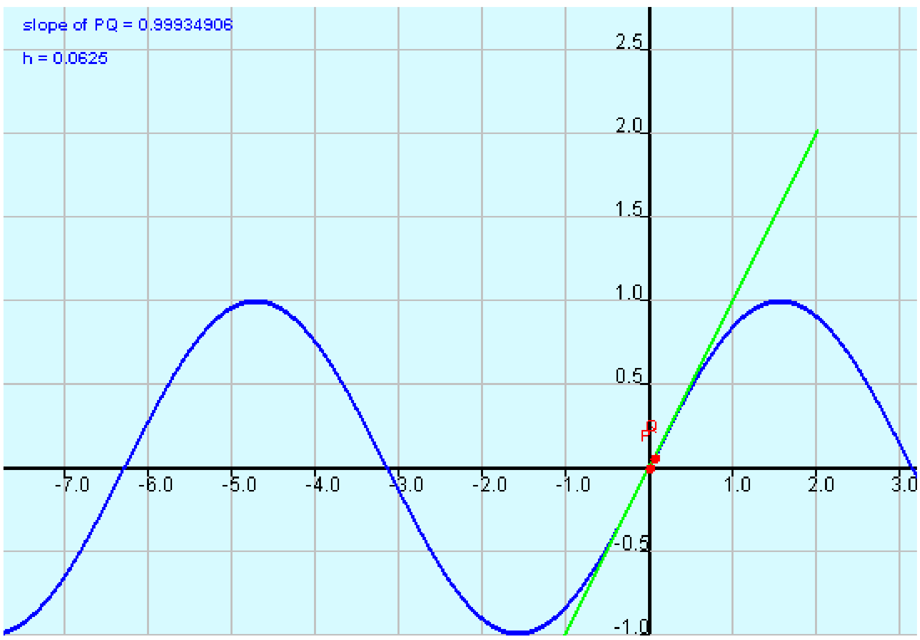
Definition. Let **y** = **f(x)** be a function. The **derivative of f** is the function whose value at x is the limit

f'

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

If this limit exists for each **x** in an open interval **I**, then we say that **f is differentiable on I**.



Notation:

$$f'(x) \Leftrightarrow \frac{dy}{dx}$$

1st derivative

Leibniz Notation

2nd derivative

$$f''(x) \Leftrightarrow \frac{d^2y}{dx^2}$$

Examples:

Use the definition of a derivative to differentiate...

(1) $f(x) = 2x^2 - 3x + 1$

(2) $y = \sqrt{x+2}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 2(x+h)^2 - 3(x+h) + 1$$

$$= 2(x^2 + 2xh + h^2) - 3x - 3h + 1$$

$$= 2x^2 + 4xh + 2h^2 - 3x - 3h + 1$$

$$\lim_{h\to 0} \frac{(\cancel{2x^2} + 4xh + 2h^2 \cancel{- 3x} - 3h \cancel{+ 1}) - (\cancel{2x^2} \cancel{- 3x} \cancel{+ 1})}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(4x + \underbrace{2h}_{\to 0} - 3)}{\cancel{h}}$$

$$f'(x) = 4x - 3$$