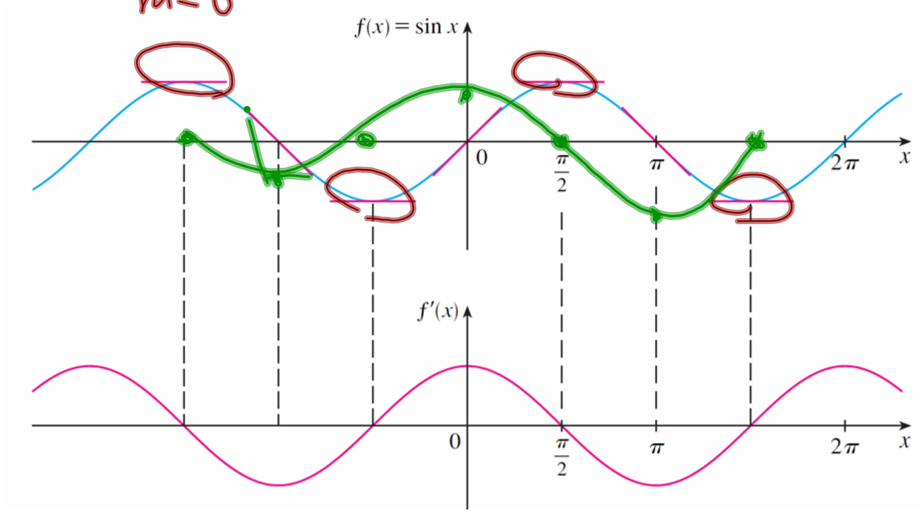
Derivatives of Trigonometric Functions

The Sine Function

- We recall that the derivative $f'(x)$ of a function $f(x)$ gives the slope of the tangent.
- On the next slide we graph $f(x) = \sin x$ together with $f'(x)$, as determined by the slope of the tangent to the sine curve.
  - Note that x is measured in radians.
- The derivative graph resembles the graph of the cosine!

$f'(x)=0$
$m=0$



$f(x) = \sin x$

Let's check this using the definition of a derivative...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Id

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right]$$

$$= \lim_{h \to 0} \sin x \cdot \overbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}^{=0} + \lim_{h \to 0} \cos x \cdot \overbrace{\lim_{h \to 0} \frac{\sin h}{h}}^{=1}$$

- Our calculations have brought us to four limits, two of which are easy:
  - Since x is constant while $h \to 0$,

    $$\lim_{h \to 0} \sin x = \sin x \quad \text{and} \quad \lim_{h \to 0} \cos x = \cos x$$

- With some work we can also show that

  $$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

- Thus our guess is confirmed:

$$f'(x) = \lim_{h \to 0} \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= (\sin x) \cdot 0 + (\cos x) \cdot 1 = \cos x$$

$f(x) = \sin x$

$f'(x) = \cos x$

$f(x) = \sin(3x^3)$

"u"

$f'(x) = \cos(\underbrace{3x^3}_{}) (9x^2)$

du

Rules to differentiate trigonometric functions:

Given that "u" represents some differentiable function...

$$\frac{d}{du}(\sin u) = \cos u \bullet du \qquad \frac{d}{du}(\csc u) = -\csc u \cot u \bullet du$$

$$\frac{d}{du}(\cos u) = -\sin u \bullet du \qquad \frac{d}{du}(\sec u) = \sec u \tan u \bullet du$$

$$\frac{d}{du}(\tan u) = \sec^2 u \bullet du \qquad \frac{d}{du}(\cot u) = -\csc^2 u \bullet du$$

Differentiate each of the following:

1. $f(x)=\cos\sqrt{5x-1}+\tan x^3$

cos "u" tan "u"

$f'(x)=-\sin\sqrt{5x-1}\left[\frac{1}{2}(5x-1)^{-1/2}(5)\right]+\sec^2 x^3(3x^2)$

2. $y=\dfrac{\sec(5x)}{\cot\sqrt{x}}$

$$y'=\frac{[\sec 5x\tan 5x(5)]\cot\sqrt{x}-\sec(5x)\left[-\csc^2\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)\right]}{\cot^2\sqrt{x}}$$

$(\csc\sqrt{x})^2$

$\cot^2\sqrt{x} \neq \cancel{\cot x}$ $(\cot\sqrt{x})^2$

3. $f(x)=\csc^2\sqrt{x}-\sqrt{\sin(9x^6)}$

$f'(x)=2(\csc\sqrt{x})^1\left(-\csc\sqrt{x}\cot\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)\right)-\frac{1}{2}(\sin 9x^6)^{-1/2}(\cos 9x^6)(54x^5)$

4. $f(x) = \tan\left[\cos\left(8x^{-3}\right)\right]$

$f'(x) = \sec^2\left[\cos\left(8x^{-3}\right)\right]\left[-\sin\left(8x^{-3}\right)\left(-24x^{-4}\right)\right]$

5. $f(x) = \sin\left\{\cos\left[\tan^3(7x)\right]\right\}$

$f'(x) = \cos\left[\cos\left(\tan^3(7x)\right)\right]\left[-\sin\left(\tan 7x\right)^3\left[3\left(\tan 7x\right)^2 \sec^2 7x\,(7)\right]\right]$

6. $y = \dfrac{6x^3\sqrt{5\cot\sqrt{x} + \cos^3 3x}}{\tan\left(\sin\sqrt[3]{x}\right) + 8\cot x^7 - \csc\left(x^4 - 1\right)^5}$

$y' = \Big[18x^2\sqrt{5\cot\sqrt{x} + \cos^3 3x} + 6x^3\left[\tfrac{1}{2}\left(5\cot\sqrt{x} + \cos^3 3x\right)^{-1/2}\left(-5\csc^2\sqrt{x}\left(\tfrac{1}{2}x^{-1/2}\right) + 3\left(\cos 3x\right)^2\left(-\sin 3x\right)(3)\right)\right]\Big]$

$\cdot\left[\tan\left(\sin\sqrt[3]{x}\right) + 8\cot x^7 - \csc\left(x^4 - 1\right)^5\right] - \left(6x^3\sqrt{5\cot\sqrt{x} + \cos^3 3x}\right)$

$\Big[\sec^2\left(\sin x^{1/3}\right)\left(\cos x^{1/3}\right)\left(\tfrac{1}{3}x^{-2/3}\right) - 8\csc^2 x^7\left(7x^6\right) + \csc\left(x^4-1\right)^5\cot\left(x^4-1\right)^5 \cdot 5\left(x^4-1\right)^4\left(4x^3\right)\Big]$

$\overline{\left(\tan\left(\sin\sqrt[3]{x}\right) + 8\cot x^7 - \csc\left(x^4 - 1\right)^5\right)^2}$

Homework

Worksheet on derivatives of trigonometric functions

Attachments

Derivatives Worksheet.doc