Advanced Math with Calculus 120

Test : Related Rates **Name:** Solutions

April 2012

1. A spherical hailstone is growing at the rate of 1 mm^3/min as it falls through the atmosphere. Determine the rate at which the radius of this hailstone is increasing when it has surface area $400\pi\, cm^2$. [8]

$V = \frac{4}{3}\pi r^3$

$SA = 4\pi r^2$

$400\pi = 4\pi r^2$

$100 = r^2$

$10 = r$

convert units $10\,cm = 100\,mm$

$10\,mm = 1\,cm$

$10^3\,mm^3 = 1\,cm^3$

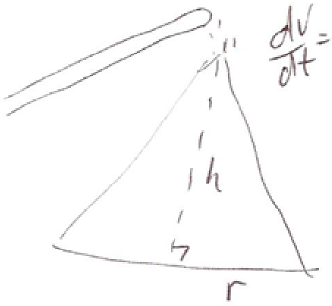
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$1 = 4\pi (100)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{40000\pi}$ mm/minute.

$= 7.96 \times 10^{-6}$ mm/minute

2. A conveyor belt system at a gravel pit pours washed sand onto the ground at the rate of 180 m^3/h. The sand forms a conical pile with height always equal to one-third the diameter of the base. Determine how fast the height of the pile is increasing at the instant when the radius at the base is 6 m. [8]



$\frac{dV}{dt} = 180\,m^3/hr.$

$h = \frac{1}{3}d$

$h = \frac{1}{3}(2r)$

$h = \frac{2}{3}r$

$r = \frac{3}{2}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h$

$V = \frac{3}{4}\pi h^3$

$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$

$180 = \frac{9}{4}\pi (4)^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \cancel{0.707\,m/hr.}$ $\frac{5}{\pi}$ m/h

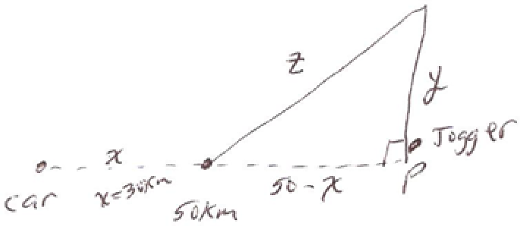
$= 1.592$ m/h

$h = \frac{2}{3}r$

$h = \frac{2}{3}(6)$

$h = 4\,m$

3. A man starts jogging northward at 10 km/h from a point P. At the same time, a car is 50 km west of P and travels on a straight road directly east (*towards P*) at 60 km/h. How fast is the distance between the jogger and the driver changing 30 minutes later? Is the distance between them increasing or decreasing? [8]



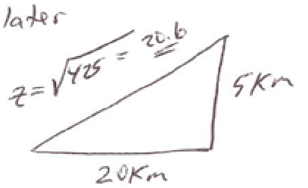
$$(50-x)^2 + y^2 = z^2$$

$$2(50-x)\left(-\frac{dx}{dt}\right) + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$2(20)(-60) + 2(5)10 = 2(\sqrt{425})\frac{dz}{dt}$$

$$-1200 + 50 = \sqrt{425}\frac{dz}{dt}$$

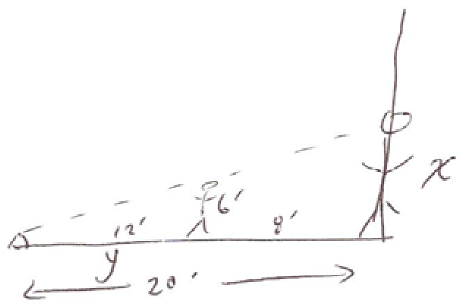
$\frac{1}{2}$hr later



$$\frac{dz}{dt} = \frac{-1150}{\sqrt{425}} = -55.78 \text{ Km/h}$$

Distance between is decreasing

4. A spot light is situated on the ground 20 ft. away from a building and a 6 ft. tall person is walking towards the building at a rate of 2.5 ft. /sec. How fast is the height of the shadow changing when the person is situated 8 feet from the wall? Is the shadow increasing or decreasing in height at this time? [8]



$$\frac{6}{y} = \frac{x}{20}$$

$$xy = 120$$

$$\frac{dx}{dt}y + x\frac{dy}{dt} = 0$$

$$\frac{dx}{dt}(12) + 10(2.5) = 0$$

$$\frac{dx}{dt} = -\frac{25}{12} \text{ feet/sec}$$

when ...

$y = 12'$

$xy = 120$

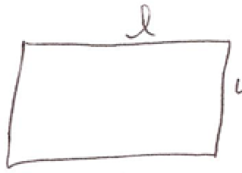
$x(12) = 120$

$x = 10$ feet

$$\frac{dx}{dt} = -2.08 \text{ feet/sec}$$

Shadow is decreasing in height

5. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at 3 cm/s. When the length of the rectangle is 20 cm and it has a perimeter of 60 cm, how fast is the area of the rectangle increasing? [8]



$\frac{dl}{dt} = 8\text{cm/sec}$

$\frac{dw}{dt} = 3\text{cm/sec}$

$A = lw$

$\frac{dA}{dt} = \frac{dl}{dt}w + l\frac{dw}{dt}$

$\frac{dA}{dt} = (8)(10) + (20)(3)$

$\frac{dA}{dt} = 140\text{cm}^2/\text{sec}$

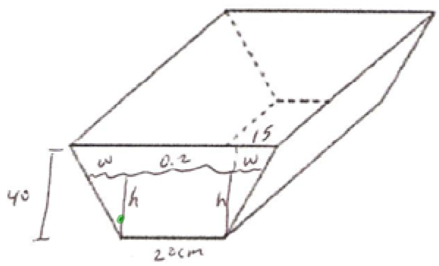
$P = 2l + 2w$

$60 = 2(20) + 2w$

$20 = 2w$

$w = 10\text{cm}$

6. A water trough is 6 m long, and its cross-section has the shape of an isosceles trapezoid that is 20 cm wide at the bottom, 50 cm wide at the top, and 40 cm deep. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 25 cm deep ? [8]



$V = \left(\frac{1}{2}wh + 0.2h + \frac{1}{2}wh\right)6$

$V = (wh + 0.2h)6$

$V = 6wh + 1.2h$

$V = 6\left(\frac{3}{8}h\right)h + 1.2h$

$V = \frac{9}{4}h^2 + 1.2h$

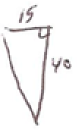
$\frac{dV}{dt} = \frac{18}{4}h\frac{dh}{dt} + 1.2\frac{dh}{dt}$

$0.2 = \frac{9}{2}(0.25)\frac{dh}{dt} + 1.2\frac{dh}{dt}$

$0.2 = 2.325\frac{dh}{dt}$

$0.086\ \text{m/minute} = \frac{dh}{dt}$

Trough



Water

$\frac{15}{w} = \frac{40}{h}$

$15h = 40w$

$w = \frac{15h}{40}$

$w = \frac{3}{8}h$

Formulas

Sphere:	***Circle:***	***Cone:***	***Prism:***
$V = \frac{4}{3}\pi r^3$	$C = 2\pi r$	$V = \frac{1}{3}\pi r^2 h$	$V = (\textit{area of face}) \times (\textit{length})$
$S.A. = 4\pi r^2$	$A = \pi r^2$		
		Cylinder:	
		$V = \pi r^2 h$	

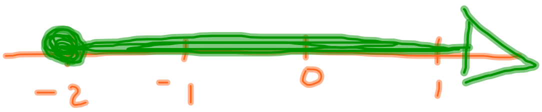
Practice Midterm

$\Rightarrow$ Omit # 3 (Not covered yet)

$\Rightarrow$ Add Complex # Question

$$(-1+i)^3(\sqrt{3}-2i)^5$$

Bracket Notation



$(-1, 2]$

$-1 < x \leq 2$

$x \geq -2$

$[-2, \infty)$

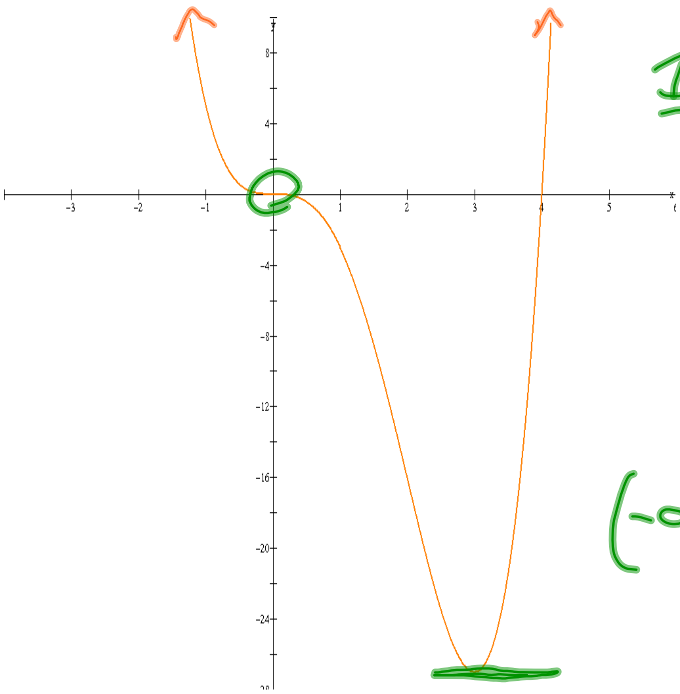
$(-\infty, 3)$

$(2, 3)$

Intervals of Increase and Decrease...

Given the function $f(x) = x^4 - 4x^3$, use the graph below to determine the intervals where *f(x)* is increasing and where *f(x)* is decreasing.

Graph...



Increasing:

$(3, \infty)$

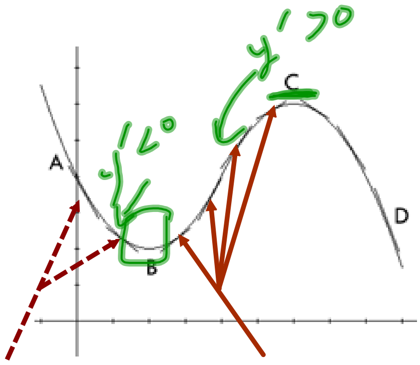
Decreasing

$(-\infty, 3)$ or

$(-\infty, 0)(0, 3)$

The Calculus of Intervals of Increase and Decrease

- Examine this graph for intervals of increase and decrease...



What do you notice about the slopes of these tangents?

What do you notice about the slopes of these tangents?

Intervals of increase?

$y' > 0$

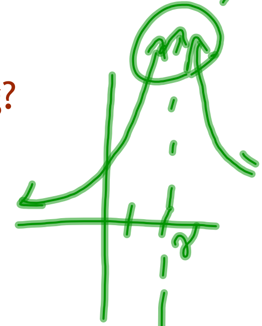
Intervals of decrease?

$y' < 0$

Critical Value(s):

$$\text{Any value of } x \text{ such that } f'(x) = 0 \text{ or } f'(x) \text{ is undefined}$$

f'(x) und



Where does *f*(*x*) switch from increasing to decreasing?
How would this tie in with Calculus?

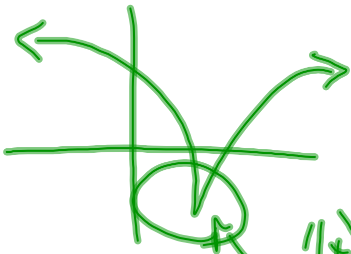
- *At the point where a function switches from increasing to decreasing, or decreasing to increasing, the derivative must be equal to 0 or undefined.*

f'(x) undefined

Let's summarize how Calculus could be used to identify regions of increase or decrease...

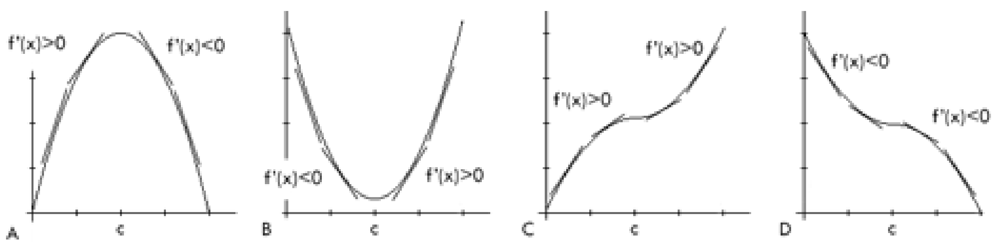
If $f'(x) > 0$ - - - - - →

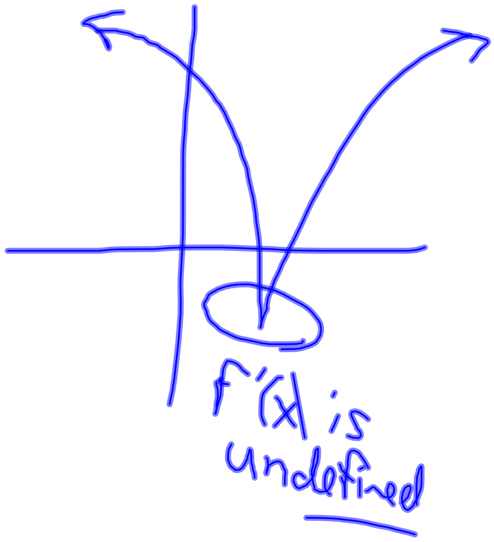
If $f'(x) < 0$ - - - - - →



f'(x) = undefined

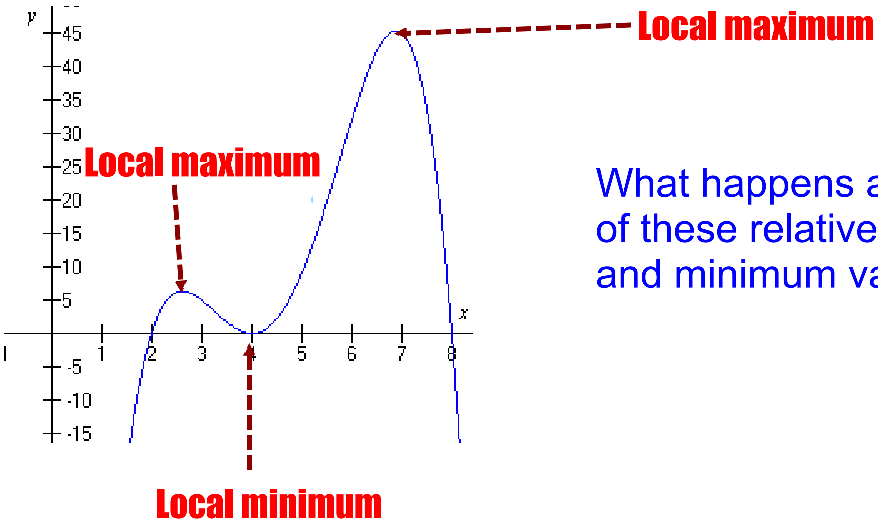
The graphs below illustrate the first derivative test.





f'(x) is undefined

Relative Extrema: (local maximum or local minimum)



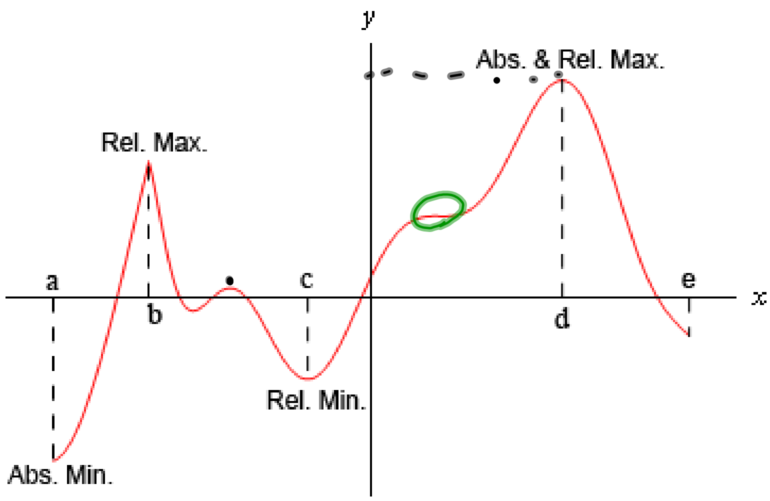
What happens at each of these relative maximum and minimum values?

We say that $f(x)$ has a **relative (or local) maximum** at $x = c$ if $f(x) \le f(c)$ for every x in some open interval around $x = c$.

- function switches from increasing to decreasing.

We say that $f(x)$ has a **relative (or local) minimum** at $x = c$ if $f(x) \ge f(c)$ for every x in some open interval around $x = c$.

- function switches from decreasing to increasing.



First Derivative Sign Table:

Intervals of increase and decrease can be organized using a first derivative sign table...

Example:

Determine the intervals of increase and decrease for the function ...

$$f(x) = x^4 - 4x^3 + 2$$

① Critical Values

$f'(x) = 4x^3 - 12x^2$

$0 = 4x^3 - 12x^2$

$0 = \underbrace{4x^2}_{+}\underbrace{(x-3)}_{-}$

$x = 0, 3$

Intervals ↓ | Factors!!

Intervals	$4x^2$	$x-3$	f'	f
$(-\infty, 0)$	+	−	−	Dec
$(0, 3)$	+	−	−	Dec
$(3, \infty)$	+	+	+	Inc

LOCAL MAX

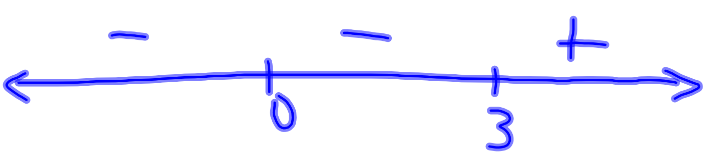
$\Rightarrow$ None

LOCAL MIN

$(3, f(3))$

$\rightarrow (3)^4 - 4(3)^3 + 2$

$f'(x)$



Another example...

Clemson University: Final Exam 2011

6. **(15 pts.)** Consider $f(x) = 6x^2 - x^3$.

a. **(7 pts.)** Identify the intervals on which $f(x)$ is increasing and decreasing. Give the ordered pair (x, y) at any local extrema, identifying each as a local maximum or local minimum. Put your answers in the blanks below.

Critical Values

$f'(x) = 12x - 3x^2$

$0 = 3x(4 - x)$

$x = 0, 4$

	$3x$	$4-x$	f'	f
$(-\infty, 0)$	$-$	$+$	$-$	Dec
$(0, 4)$	$+$	$+$	$+$	Inc
$(4, \infty)$	$+$	$-$	$-$	Dec

Local Max

$(4, f(4))$

$(4, 32)$

Local Min

$(0, 0)$