Sometimes long division and completing the square can help...

Here is another tricky one....

$$\int \sec x dx =$$

$$\cancel{1+\tan^2 x = \sec^2 x}$$

$$\cancel{\sec x = \sqrt{1+\tan^2 x}}$$

$$\int \frac{\cos x}{\cos^2 x} dx$$

$$\int \frac{\cos x}{1-\sin^2 x} dx$$

derivative

$$-2(\sin x)^1(\cos x)$$

$$\left.\begin{aligned} u &= \sin x \\ du &= \cos x dx \end{aligned}\right\} = \int \frac{du}{1-u^2}$$

Partial Fractions

$$\frac{A}{1-u} + \frac{B}{1+u} = \frac{1}{1-u^2}$$

$$\int \frac{1 du}{(1-u)(1+u)}$$

$$A(1+u) + B(1-u) = 1$$

$$A + Au + B - Bu = 1$$

$$A+B=1 \qquad A-B=0$$

$$A+B=1$$

$$2A=1$$

$$A=\frac{1}{2}$$

$$\frac{1}{2}+B=1$$

$$B=\frac{1}{2}$$

$$= \frac{-1}{2}\int \frac{(-1)\,du}{1-u} + \frac{1}{2}\int \frac{du}{1+u}$$

$$= \frac{-1}{2}\ln(1-u) + \frac{1}{2}\ln(1+u) + C$$

$$= \frac{-1}{2}\ln|1-\sin x| + \frac{1}{2}\ln|1+\sin x| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$\frac{1}{2}\left(\ln|1+\sin x| - \ln|1-\sin x|\right)$$

$$\sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x} \; .$$

ry the substitution $u = \sin x,\ du = \cos x dx$. Then

$$\int \sec x dx = \int \frac{du}{1 - u^2} \; .$$

e a dead end, but a little algebra pulls us through. The identity

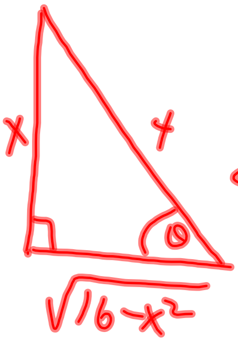
$$\frac{1}{1 - u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)$$

$$\int \frac{du}{1-u^2} dx = \frac{1}{2}\int\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du = \frac{1}{2}\left(\ln(1+u) - \ln(1-u)\right) + C$$

Warm Up

Evaluate each of the following:

$$\int \frac{1}{(16-x^2)^{\frac{3}{2}}}\,dx$$ (UNB: 2005)



$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

$4\cos\theta = \sqrt{16-x^2}$

$4\sin\theta = x$

$4\cos\theta\, d\theta = dx$

$\int \frac{4\cos\theta\, d\theta}{(4\cos\theta)^3}$

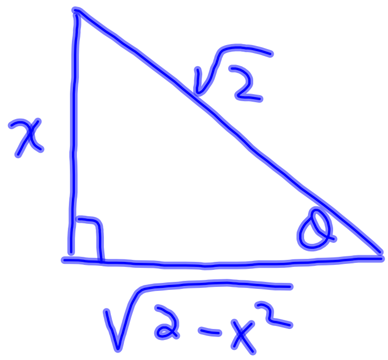
$\frac{1}{16}\int \frac{d\theta}{\cos^2\theta}$

$\frac{1}{16}\int \sec^2\theta\, d\theta$

$= \frac{1}{16}\tan\theta + C$

$= \frac{1}{16}\frac{x}{\sqrt{16-x^2}} + C$

$$\int \frac{\sqrt{2-x^2}}{x}\,dx$$ (UNB: 2004)



$\sqrt{2}\cos\theta = \sqrt{2-x^2}$

$\sqrt{2}\sin\theta = x$

$\sqrt{2}\cos\theta\, d\theta = dx$

$\int \frac{\sqrt{2}\cos\theta\,\sqrt{2}\cos\theta\, d\theta}{\sqrt{2}\sin\theta}$

$\frac{2}{\sqrt{2}}\int \frac{\cos^2\theta}{\sin\theta}\, d\theta$

$$\int (\csc^2 t \cot^2 t)\, dt$$ (UNB: 2004)

$-\int (\cot t)^2 (-\csc^2 t)\, dt$

$-\frac{1}{3}\cot^3 t + C$

$$\int \frac{\ln(\ln x)}{x}\, dx$$ (StFX: 2005)

$u = \ln x$

$du = \frac{1}{x} dx$

$\int \ln u \, du$

$w = \ln u \qquad dv = du$

$dw = \frac{1}{u} du \qquad v = u$

$= u\ln u - \int u \frac{1}{u} du$

$= u\ln u - \int du$

$= u\ln u - u + C$

$= \ln x \ln(\ln x) - \ln x + C$

Integration using Partial Fractions

Simplify: $\frac{3}{x-5}+\frac{2}{x+4}$

$$=\frac{3(x+4)+2(x-5)}{(x-5)(x+4)}$$

$$=\frac{5x+2}{(x-5)(x+4)}$$

We want to reverse the process of
finding a common denominator...

Express as partial fractions $\int \frac{5x+2}{x^2-x-20}\,dx$

1. Factor the denominator: $\frac{5x+2}{(x-5)(x+4)}$

2. Separate into partial fractions:

$$\frac{A}{x-5}+\frac{B}{x+4}=\frac{5x+2}{x^2-x-20}$$

3. Find common denominator and solve for A and B:

$$\frac{A(x+4)+B(x-5)}{(x-5)(x+4)}=\frac{5x+2}{x^2-x-20} \qquad \frac{7}{3}=\frac{}{3}$$

$$Ax+4A+Bx-5B=5x+2$$

$A+B=5$ $\qquad 4A-5B=2$

$A=5-B$ $\qquad 4(5-B)-5B=2$

$A=5-2$ $\qquad 20-4B-5B=2$

$A=3$ $\qquad 20-9B=2$

$-9B=-18$

$B=2$

$$\int \frac{5x+2}{x^2-x-20}\,dx=\int\left(\frac{3}{x-5}+\frac{2}{x+4}\right)dx$$

$$3\int\frac{dx}{x-5}+2\int\frac{dx}{x+4}$$

$$=3\ln|x-5|+2\ln|x+4|+C$$

Now let's evaluate the following integral...

$$\int \frac{(5x+2)dx}{x^2 - x - 20}$$

Here is another example...

$$\int \frac{xdx}{x^2 - 3x + 2}$$

Some special situations involving partial fractions...

Note 1:

- If the degree of the numerator is the same as that of the denominator, or higher, we would have to take the preliminary step of first performing a long division.

Ex. $$\frac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1}$$

$3x^6y^7 \rightarrow$ degree 13

$$\begin{array}{r} x - 6 \\ 2x^2 + x - 1 \overline{\smash{\big)}\, 2x^3 - 11x^2 - 2x + 2} \\ \underline{2x^3 + x^2 - x} \end{array} = (x-6) + \frac{5x-4}{(x+1)(2x-1)}$$

Note 2:

- If the denominator has more than two linear factors, we must include a term corresponding to each factor.

Ex. $$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}$$

Note 3:

- If a linear factor is repeated, we need to include extra terms in the partial fraction expression.

Ex. $$\frac{x}{(x+3)^2(x-2)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-2}$$

Note 4:

- When we factor the denominator as far as possible, it may happen that we end up with an irreducible quadratic factor of the form $ax^2 + bx + c$, where the discriminant is negative. Then the corresponding partial fraction is of the form...

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. This term can be integrated by completing the square and by using the integration formula...

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$ ***(or a trig. substitution)***