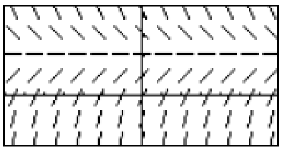
Match the slope fields with their differential equations.

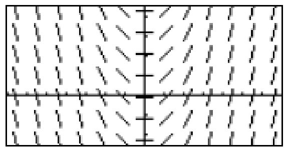
(A)

9



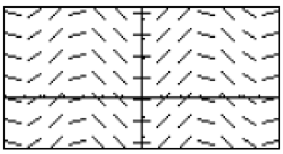
(B)

10



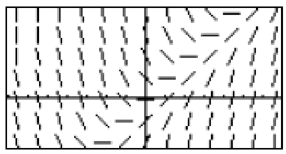
(C)

7



(D)

8



7. $\dfrac{dy}{dx} = \sin x$

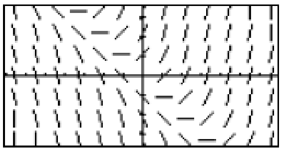
8. $\dfrac{dy}{dx} = x - y$

9. $\dfrac{dy}{dx} = 2 - y$

10. $\dfrac{dy}{dx} = x$

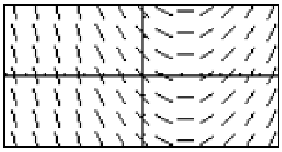
Match the slope fields with their differential equations.

(A)



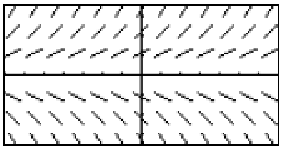
14

(B)



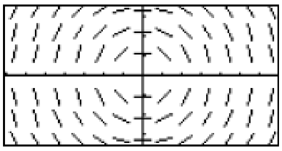
11

(C)



12

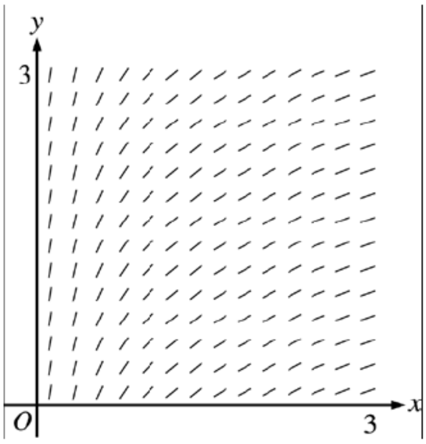
(D)



13

11. $\frac{dy}{dx} = 0.5x - 1$ 12. $\frac{dy}{dx} = 0.5y$ 13. $\frac{dy}{dx} = -\frac{x}{y}$ 14. $\frac{dy}{dx} = x + y$

From the May 2008 *AP Calculus Course Description:*
15.



The slope field from a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

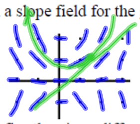
y' = sin x

(A) $y = x^2$ (B) $y = e^x$ (C) $y = e^{-x}$ (D) $y = \cos x$ (E) $y = \ln x$

$y' = 2x$ $y' = e^x$ $y' = -e^{-x}$ $y' = \frac{1}{x}$

17. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

(A) On the axes provided, sketch a slope field for the given differential equation.



(B) Let f be the function that satisfies the given differential equation. Write an equation for the tangent line to the curve $y = f(x)$ through the point (1, 1). Then use your tangent line equation to estimate the value of $f(1.2)$.

(C) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(1) = 1$. Use your solution to find $f(1.2)$.

(D) Compare your estimate of $f(1.2)$ found in part (b) to the actual value of $f(1.2)$ found in part

(E) Was your estimate from part (b) an underestimate or an overestimate? Use your slope field to explain why.

B) $\frac{dy}{dx} = \frac{xy}{2}$ at $(1,1)$

$m = \frac{(1)(1)}{2} = \frac{1}{2}$

$y - 1 = \frac{1}{2}(x-1)$

$y = \frac{1}{2}x - \frac{1}{2} + 1$

$y = \frac{1}{2}x + \frac{1}{2}$

$f(1.2) = \frac{1}{2}(1.2) + \frac{1}{2}$

$\doteq 1.1$

C/ $\frac{dy}{dx} = \frac{xy}{2}$ $(1,1)$

$\int \frac{1}{y}\,dy = \int \frac{x}{2}\,dx$

$\ln y = \frac{1}{2}\left(\frac{x^2}{2}\right) + C$ $(1,1)$

$\ln 1 = \frac{(1)^2}{4} + C$

$0 = \frac{1}{4} + C$

$-\frac{1}{4} = C$

$\ln y = \frac{x^2}{4} - \frac{1}{4}$

$e^{\frac{x^2-1}{4}} = y$

$f(x) = e^{\frac{x^2-1}{4}}$

$f(1.2) = e^{\frac{(1.2)^2-1}{4}} \doteq 1.12$

E/ estimate ⇒ 1.1

actual ⇒ 1.12

∴ underestimate

Concave Up through (1,1)

∴ underestimate

Exponential Growth and Decay

Law of Exponential Change (Law of Natural Growth)

Exponential growth problems involve growth in which the rate of change is proportional to the amount present.

ie. The more bacteria in a petri dish, the faster they multiply
The more radioactive material present, the faster it decays
The more money in your bank account, the faster it grows (assuming compound interest)

The differential equation that describes this type of growth is…

$\frac{dy}{dt} = ky \quad (k > 0)$, where k is the growth constant (if positive) or the decay constant (if negative).

This differential equation can be solved by separating the variables:

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}dy = kdt$$ ← Separate Variables

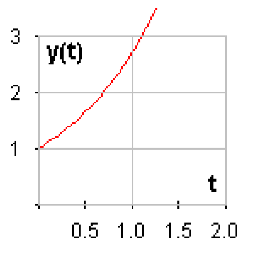
$$\ln|y| = kt + C$$ ← Antidifferentiate both sides

$$|y| = e^{kt+C}$$ ← Convert to exponential form

$$|y| = e^{C}e^{kt}$$ ← Laws of exponents

$$y = \pm e^{C}e^{kt}$$ ← Definition of absolute value

$$y = Ae^{kt}$$ ← Let $A = \pm e^{C}$

INITIAL VALUE PROBLEM	SOLUTION	GRAPH
$\frac{dy}{dt} = ky$ $y(0) = y_o$	$y(t) = y_o e^{kt}$	

$$\frac{dy}{dt} = ky \quad (k>0)$$

$$\int \frac{1}{y}\,dy = \int k\,dt$$

$$\ln|y| = kt + C$$

$$e^{kt+C} = |y|$$

$$y = (e^{kt})(e^{C})$$

$$y = (\pm e^{C})e^{kt}$$

$A = \pm e^{C}$ } This is some constant Value

"Formula:"

$$y = Ae^{kt}$$

Initial Amount

$$y(0) = A$$

Notice that f(0)=A , so this implies that the constant A is the size of the quantity at t = 0.

Example:

The population of bacteria grown in a culture follows the law of natural growth, with a growth rate of 15% per hour. If there are 10 000 bacteria present initially, how many will there be after 4 hours?

Let P denote the number of bacteria present at time t :

$$\frac{dP}{dt} = kP$$

$$\frac{dp}{dt} = 0.15P$$

$$P = Ae^{0.15t} \text{ ,when t } = 0, \text{P} = 10\,000, \text{so}$$

$$10\ 000 = Ae^{0} \text{ ,hence A} = 10\,000$$

$$\therefore P = 10000e^{0.15t}$$

$$\textit{substitute } t = 4...$$

$$P = 10000e^{0.15(4)} \approx 18000$$

Example:

According to United Nations data, the world population at the beginning of 1990 was approximately 5.3 billion and growing at a rate of 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2015.

(solution: 8.7 billion)

$$\frac{dP}{dt} = 0.02P$$

$$P = Ae^{0.02t}$$

$$P = 5.3e^{0.02t}$$

$\therefore$ at $t = 25$

$$P = 5.3e^{0.02(25)}$$

$$\underline{P \doteq 8.7 \text{ billion}}$$

More Examples...

1. Assume that the population of the U.S. increases at a rate proportional to the population, and that the population was 150 million in 1950 and 200 million in 1970.
   a.) Estimate the population for the year 2000.

   b.) Approximately when will the population be one billion?

$$\frac{dP}{dt} = kP$$

$t = 0$ (1950) $\Rightarrow A = 150$ million

$t = 20$ (1970)

$$P = Ae^{kt}$$

$$\therefore P = 150e^{kt}$$

$$200 = 150e^{k(20)}$$

$$\frac{200}{150} = e^{20k}$$

$$\ln\left(\frac{4}{3}\right) = \ln e^{20k}$$

$$\ln\frac{4}{3} = 20k$$

$$k = \frac{\ln\left(\frac{4}{3}\right)}{20} \longrightarrow P = 150(e)^{\frac{\ln\left(\frac{4}{3}\right)}{20}t}$$

2000 ($t = 50$)

$$P = 150e^{\left[\frac{\ln\frac{4}{3}}{20}(50)\right]}$$

$P = 308$ million

$$\frac{1\,000\,000\,000}{1\,000\,000}$$

$$\frac{10^9}{10^6} = 10^3$$

b)

$$1000 = 150e^{\frac{\left(\ln\frac{4}{3}\right)}{20}t}$$

$$\frac{1000}{150} = e^{\frac{\ln\left(\frac{4}{3}\right)}{20}t}$$

$$\ln\left(\frac{1000}{150}\right) = \frac{\ln\left(\frac{4}{3}\right)}{20}t$$

$$\frac{20\ln\left(\frac{1000}{150}\right)}{\ln\left(\frac{4}{3}\right)} = t$$

$t = 132$ years