Warm Up

Solve the following separable differential equations:

$$2xy^2\frac{dy}{dx} = y^2 + 1$$

$$\int \frac{y^2}{y^2+1}\,dy = \int \frac{1}{2x}\,dx$$

$$y^2+1 \overline{\big) y^2} \quad 1$$
$$y^2+1$$
$$-1$$

$$\int \left(1 - \frac{1}{y^2+1}\right)dy = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$y - \tan^{-1} y + C = \frac{1}{2}\ln|x|$$

$$(\ln x)^4$$

$$\frac{dy}{dx} = \frac{4\sqrt{y}\ln x}{x},\ y(e) = 1$$

$$\frac{1}{\sqrt{y}}\,dy = \frac{4\ln x}{x}\,dx$$

$$\int y^{-1/2}\,dy = 4\int \frac{\ln x}{x}\,dx$$

$$2y^{1/2} + C = \frac{4(\ln x)^2}{2}$$

$x = e$
$y = 1$

$$2(1)^{1/2} + C = \frac{4(\ln e)^2}{2}$$

$$2 + C = 4/2$$

$$C = 0$$

$$2\sqrt{y} = 2\ln^2 x$$

$$(\sqrt{y})^2 = (\ln^2 x)^2$$

$$y = \ln^4 x$$

2. One hundred fruit flies are placed in a breeding container that can support a population of at most 5000 fruit flies. If the population grows exponentially at the rate of 2% per day, how long will it take for the container to reach capacity?

$$N = N_0 e^{kt}$$

$$5000 = 100e^{0.02t}$$

$$50 = e^{0.02t}$$

$$\ln 50 = \ln e^{0.02t}$$

$$\ln 50 = 0.02t$$

$$t = \frac{\ln 50}{0.02}$$

$t \doteq 196$ days

$$\frac{dy}{dt} = ky$$

$$\underline{\underline{k = 0.02}}$$

days

t	0	1
N	100	102

$\times 1.02$

$$N = 100(1.02)^t$$

$$5000 = 100(1.02)^t$$

$$50 = 1.02^t$$

$$\ln 50 = \ln 1.02^t$$

$$t = \frac{\ln 50}{\ln 1.02}$$

$$t = 198$$

3. A radioactive substance decays at a rate proportional to the amount present. Assuming the "half life" is 5 years, how onlg will it be until only 1% of the original substance remains?

$$A = A_0 e^{kt}$$

$$\frac{1}{2} = 1e^{k(5)}$$

$$\ln\frac{1}{2} = \ln e^{5k}$$

$$\ln\frac{1}{2} = 5k$$

$$k = \frac{\ln\frac{1}{2}}{5}$$

$$A = A_0 e^{\frac{\ln\frac{1}{2}}{5}t}$$

$$0.01\cancel{A_0} = \cancel{A_0} e^{\frac{\ln 1/2}{5}t}$$

$$\ln 0.01 = \ln e^{\frac{\ln 1/2}{5}t}$$

$$\ln 0.01 = \frac{\ln 1/2}{5} t$$

$$\frac{5\ln(0.01)}{\ln 1/2} = t$$

$$t \doteq 33 \text{ years}$$

t	0	5
A	M_0	$\frac{1}{2}M_0$

$$M = M_0\left(\frac{1}{2}\right)^{t/5}$$

$$0.01 = 1\left(\frac{1}{2}\right)^{t/5}$$

$$\ln 0.01 = \ln\left(\frac{1}{2}\right)^{t/5}$$

$$\ln(0.01) = \frac{t}{5}\ln\left(\frac{1}{2}\right)$$

4. A certain substance decomposes at a rate proportional to its weight. If 9 grams ofthe substance are initially present and 1 gram decomposes in the first hour, how long will it take before 5 grams have decomposed?

$$m = m_0 e^{kt}$$

Mass Remaining

$$8 = 9e^{k(1)}$$

$$\frac{8}{9} = e^{k}$$

$$\ln\left(\frac{8}{9}\right) = k$$

$$m = m_0 e^{\left(\ln \frac{8}{9}\right)t}$$

$$4 = 9e^{\left(\ln 8/9\right)t}$$

$$\ln\frac{4}{9} = \ln(8/9)\,t$$

$$t = \frac{\ln\left(\frac{4}{9}\right)}{\ln(8/9)}$$

$$t \doteq 7 \text{ hours}$$

Newton's Law of Cooling

The rate at which a hot body cools to the temperature of its surroundings is proportional to the temperature difference between the body and its surroundings.

ie. A cup of tea at 60 degrees above its surroundings cools degree by degree twice as rapidly as when it is 30 degrees above its surroundings.

****Note: This law also applies to warming***

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

body's temp (changing)

$\frac{dy}{dt} = ky$

$$\frac{dT}{dt} = k(T - T_s)$$

Surrounding Temp. (constant)

where k is a constant. This could be solved as a separable differential equation, but an easier method would be to change variables...

Let $y(t) = T(t) - T_s$

$$y(t) = T(t) - T_s$$

$$y'(t) = T'(t) - 0$$

$$\therefore \frac{dy}{dt} = \frac{dT}{dt}$$

This means that our initial differential equation can be expressed as...

$$\frac{dy}{dt} = ky$$

Note: If the object is cooling k must be negative because the temperature is decreasing as time passes.

$\frac{dy}{dt} = ky$

$y = Ae^{kt}$

$$\frac{dT}{dt} = K(\overbrace{T - T_s}^{\text{"}y\text{"}})$$

$$\boxed{T(t) - T_s = Ae^{kt}} \Longrightarrow A = \text{Initial Temperature Difference}$$

Example:

A cup of water with a temperature of 95ºC is placed in a room with a constant temperature of 21ºC.
(a) Assuming Newton's Law of Cooling applies, set up and solve an initial-value problem whose solution is the temperature of the water *t* minutes after it is placed in the room.
(b) How many minutes will it take for the water to reach a temperature of 51ºC if it cools to 85ºC in 1 minute?

(a) $\frac{dT}{dt} = k(T - T_s)$

Let $y(t) = T(t) - T_s$

then $\frac{dy}{dt} = ky$, where $y(0) = 95°C - 21°C = 74°C$

$A = T_i - T_s$

$\therefore y = Ae^{kt}$

$74 = Ae^0$, $\therefore A = 74$

$y = 74e^{kt}$ now re-substitute $y(t) = T(t) - T_s$

$T(t) - T_s = 74e^{kt}$

$T(t) = 74e^{kt} + 21$ *or* $T(t) = 74e^{-kt} + 21$(decreasing)

Newton's Law of Cooling	Solution
$\frac{dT}{dt} = -k(T - T_a).$ $T(0) = T_o$	$T(t) = T_a + (T_o - T_a)e^{-kt}$

(b) $T(t) = 74e^{kt} + 21$ Must first determine the proportionality constant

$T(1) = 74e^k + 21 = 85$

$\therefore 74e^k = 64$

$e^k = \frac{64}{74}$

$k = \ln\frac{64}{74} = -0.145$

$T(t) = 74e^{-0.145t} + 21$

$51 = 74e^{-0.145t} + 21$

$30 = 74e^{-0.145t}$

$e^{-0.145t} = \frac{30}{74}$ $\therefore -0.145t = \ln\frac{30}{64}$

$t = 6.227$ minutes