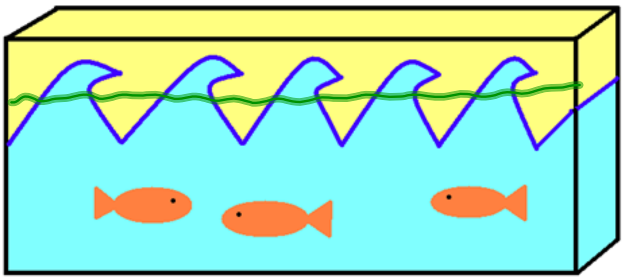
Average Value of a Function

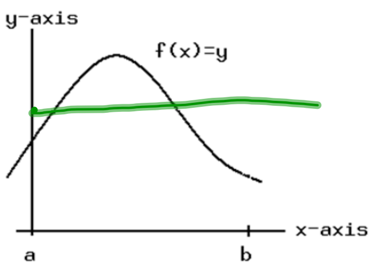
Consider the following picture:



- How high would the water level be if the waves all settled?

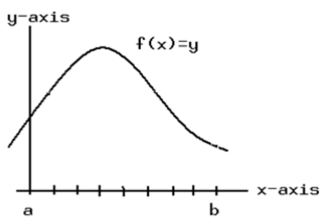
...Would it have anything to do with average??

Suppose we have a “nice”function and we need to find its average value over the interval [a,b].



Let’s apply our knowledge of how to find the average over a finite set of values to this problem:

First, we partition the interval [a,b] into n subintervals of equal length to get back to the finite situation:



In the above graph, we have n=8

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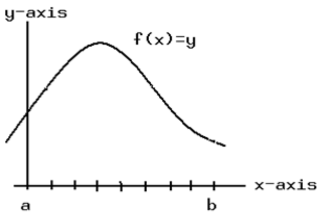

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Try this, to *approximate* the average:

- Split $[a, b]$ into subintervals width $\Delta x = (b - a)/n$.
- Pick a point x_i in each subinterval.
- Average just the $f(x_i)$:

$$= \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

estimate for the average value

As n increases, the approximation gets better.

Sounds and looks a bit like an integral!

Now, get Δx into the mix:

$$\text{Average} \approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

Multiply *and* divide by Δx, so nothing changes.

$$= \frac{\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\Delta x}{n\,\Delta x}$$

Use the fact that $\Delta x = (b - a)/n$, or $n\,\Delta x = (b - a)$.

$$= \frac{\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)\Delta x}{(b - a)}$$

Top becomes an integral as n increases…

In a more condensed form, we now get:

$$average \approx \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

But we want to get out of the finite, and into the infinite!

How do we do this?

Take Limits!!!

In this way, we get the average value of f(x) over the interval [a,b]:

$$average = \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx$$

So, if f is a "nice" function (i.e. we can compute its integral) then we have a precise solution to our problem.

Average Value of a Function

So we make the following definition: The average value of a function $f(x)$ on an interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Example

Find the average value of the function $f(x) = x^2$ on the interval $[-1, 1]$.

$= \frac{1}{3}$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

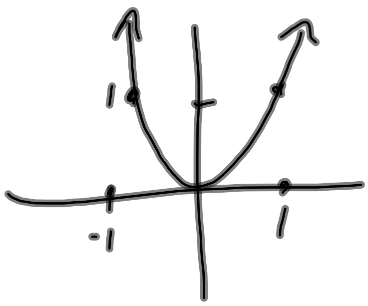
$$\frac{1}{2}\int_{-1}^{1} x^2\,dx$$

$$\frac{1}{2}\left(\frac{x^3}{3}\right)\Bigg|_{-1}^{1}$$

$$\frac{1}{2}\left[\frac{1}{3} - \left(\frac{-1}{3}\right)\right]$$

$$\frac{1}{2}\left(\frac{2}{3}\right)$$

$$= \frac{1}{3}$$



Example

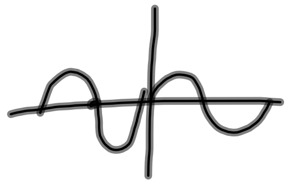
$$f(t)=t^2-5t+6\cos(\pi t) \text{ on } \left[-1,\frac{5}{2}\right]$$

$= -1.620993$

$\frac{5}{2}+1=\frac{7}{2}$

$$\frac{1}{\left(\frac{7}{2}\right)}\int_{-1}^{5/2} t^2-5t+\frac{6}{\pi}\cos(\pi t)(\pi)\,dt$$

$$=\frac{2}{7}\left[\frac{t^3}{3}-\frac{5}{2}t^2+\frac{6}{\pi}\sin(\pi t)\right]\Bigg|_{-1}^{5/2}$$



$$=\frac{2}{7}\left[-\left(-\frac{1}{3}-\frac{5}{2}+\frac{6}{\pi}\sin(-\pi)\right)+\left(\frac{125}{24}-\frac{125}{8}+\frac{6}{\pi}\sin\frac{5\pi}{2}\right)\right]$$

$\sin(-\pi)=0$, $\sin\frac{5\pi}{2}=1$

$$=\frac{2}{7}\left[\frac{125}{24}-\frac{125}{8}+\frac{6}{\pi}+\frac{1}{3}+\frac{5}{2}\right]=-1.621$$

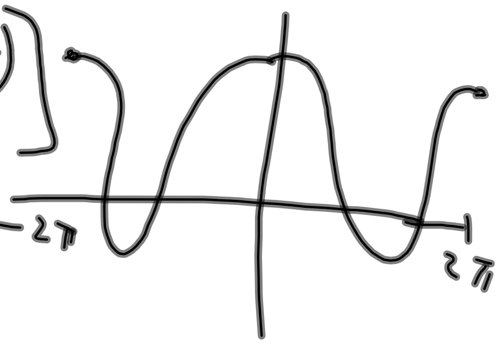
Example

$R(z) = \sin(2z)\,e^{1-\cos(2z)}$ on $[-\pi, \pi]$ $= 0$

$$\left(\frac{1}{2}\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{1-\cos 2z}(\sin 2z)(2)\,dz$$

$$\frac{1}{4\pi}\left(e^{1-\cos 2z}\right)\Big|_{-\pi}^{\pi}$$

$$\frac{1}{4\pi}\left[e^{1-\cos 2\pi} - e^{1-\cos(-2\pi)}\right]$$



$$\frac{1}{4\pi}\left(e^{1-1} - e^{1-1}\right)$$

$$= \frac{1}{4\pi}(1-1)$$

$$= \underline{\underline{0}}$$