Here is another neat trick...

$$\int \sec x dx =$$

$$\int \sec x \, dx \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right)$$

$$\int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx$$

$$= \ln|\tan x + \sec x| + C$$

Hint 1...here is what the answer will look like with this trick

$$\ln|\sec \cancel{x} + \tan \cancel{x}| + C$$

Hint 2...It is always legal to multiply by 1

$$d(\ln|\sec x + \tan x|)$$

$$= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)$$

$$\int \csc x\,dx \left(\frac{\csc x+\cot x}{\csc x+\cot x}\right)$$

$$-1\int \frac{-\csc^2 x - \csc x\cot x}{\csc x+\cot x}\,dx$$

$$= -\ln\left|\csc x+\cot x\right| + C$$

Warm Up

Evaluate each of the following:

$$\int \frac{1}{(16-x^2)^{\frac{3}{2}}}\,dx$$ (UNB: 2005)

$$\int \frac{\sqrt{2-x^2}}{x}\,dx$$ (UNB: 2004)

$$\int \left(\csc^2 t \cot^2 t\right)\,dt$$ (UNB: 2004)

$$\int \frac{\ln(\ln x)}{x}\,dx$$ (StFX: 2005)

$$\frac{2}{\sqrt{2}}\int \frac{\cos^2\theta}{\sin\theta}\,d\theta$$

$$\frac{2}{\sqrt{2}}\int \frac{1-\sin^2\theta}{\sin\theta}\,d\theta$$

$$\frac{2}{\sqrt{2}}\int (\csc\theta - \sin\theta)\,d\theta$$

$$\frac{2}{\sqrt{2}}\left[\int \csc\theta\,d\theta - \int \sin\theta\,d\theta\right]$$

$$\frac{2}{\sqrt{2}}\left[\int \csc\theta\,d\theta\left(\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}\right) + \cos\theta\right] + C$$

$$\frac{2}{\sqrt{2}}\left[-\int \frac{\csc^2\theta + \csc\theta\cot\theta}{\cot\theta + \csc\theta}\,d\theta + \cos\theta\right] + C$$

$$\frac{2}{\sqrt{2}}\left(-\ln|\csc\theta + \cot\theta| + \cos\theta\right) + C$$

Integration using Partial Fractions

Simplify: $\frac{3}{x-5}+\frac{2}{x+4}$

We want to reverse the process of
finding a common denominator...

Express as partial fractions: $\frac{5x+2}{x^2-x-20}$

1. Factor the denominator: $\frac{5x+2}{(x-5)(x+4)}$

2. Separate into partial fractions:

$$\frac{A}{x-5}+\frac{B}{x+4}=\frac{5x+2}{x^2-x-20}$$

3. Find common denominator and solve for A and B:

Now let's evaluate the following integral...

$$\int \frac{(5x+2)dx}{x^2 - x - 20}$$

Here is another example...

$$\int \frac{xdx}{x^2 - 3x + 2}$$

Some special situations involving partial fractions...

Note 1:

- If the degree of the numerator is the same as that of the denominator, or higher, we would have to take the preliminary step of first performing a long division.

Ex. $\dfrac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1}$

$$2x^2 + x + 1\overline{)2x^3 - 11x^2 - 2x + 2} = (x-6) + \frac{5x-4}{(x+1)(2x-1)}$$

Note 2:

- If the denominator has more than two linear factors, we must include a term corresponding to each factor.

Ex. $\dfrac{x+6}{x(x-3)(4x+5)} = \dfrac{A}{x} + \dfrac{B}{x-3} + \dfrac{C}{4x+5}$

$\frac{1}{a^2} + \frac{1}{a}$ $\quad \frac{3}{x+2} - \frac{4}{(x+2)^2} + \frac{6}{x} = \frac{\quad}{x(x+2)^2}$

Note 3:

- If a linear factor is repeated, we need to include extra terms in the partial fraction expression.

Ex. $\dfrac{x}{(x+3)^2(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2} + \dfrac{C}{x-2}$

Note 4:

- When we factor the denominator as far as possible, it may happen that we end up with an irreducible quadratic factor of the form $ax^2 + bx + c$, where the discriminant is negative. Then the corresponding partial fraction is of the form...

$\frac{Ax+B}{x^2+2}$ $\quad \dfrac{Ax+B}{ax^2+bx+c} \quad$ $\frac{Ax^2+Bx+C}{ax^3+bx^2+cx+d}$

where A and B are constants to be determined. This term can be integrated by completing the square and by using the integration formula...

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \textit{(or a trig. substitution)}$$

$$2x^2+x-1\overline{)\,2x^3-11x^2-2x+2\,} = (x-6)+\frac{5x-4}{(x+1)(2x-1)}$$

$x-6$

$2x^3+x^2-x$

$-12x^2-x+2$

$-12x^2-6x+6$

$5x-4$

$2x^2+x-1$

$2x^2+2x-x-1$

$2x(x+1)-1(x+1)$

$(2x-1)(x+1)$

Ex. $\displaystyle\int\left(\frac{2x^3-11x^2-2x+2}{2x^2+x-1}\right)dx$

$$\int (x-6)+\frac{5x-4}{(2x-1)(x+1)}\,dx$$

$$\int(x-6)\,dx+\int\frac{5x-4}{(2x-1)(x+1)}\,dx$$

Partial Fractions

$$\frac{A}{2x-1}+\frac{B}{x+1}=\frac{5x-4}{(2x-1)(x+1)}$$

$A(x+1)+B(2x-1)=5x-4$

$Ax+A+2Bx-B=5x-4$

$A+2B=5$ $\quad A-B=-4$

$A-B=-4$

$3B=9$

$B=3$

$A-3=-4$

$A=-1$

$$\int(x-6)\,dx+\frac{1}{2}\int\frac{-1(2)}{2x-1}\,dx+\int\frac{3}{x+1}\,dx$$

$$=\frac{x^2}{2}-6x-\frac{1}{2}\ln|2x-1|+3\ln|x+1|+C$$

Ex. $\dfrac{x+6}{x(x-3)(4x+5)} = \dfrac{A}{x} + \dfrac{B}{x-3} + \dfrac{C}{4x+5}$

$A(x-3)(4x+5) + B(x)(4x+5) + C(x)(x-3) = x+6$

$A(4x^2-7x-15) + B(4x^2+5x) + C(x^2-3x) = x+6$

$4Ax^2 - 7Ax - 15A + 4Bx^2 + 5Bx + Cx^2 - 3Cx = x+6$

① $4A + 4B + C = 0$ ② $-7A + 5B - 3C = 1$ ③ $-15A = 6$

$-\frac{8}{5} + 4B + C = 0$ $\frac{14}{5} + 5B - 3C = 1$ $A = \frac{-6}{15} = -\frac{2}{5}$

$\times 3 \left(4B + C = \frac{8}{5}\right)$ $5B - 3C = -\frac{9}{5}$

$12B + 3C = \frac{24}{5}$

$5B - 3C = -\frac{9}{5}$

$17B = 3 \longrightarrow 4\left(\frac{3}{17}\right) + C = \frac{8}{5}$

$B = \frac{3}{17}$

$C = \frac{8}{5} - \frac{12}{17}$

$C = \frac{76}{85}$

Examples...

(a) $\displaystyle\int \frac{x-1}{x(x^2+2x+1)}\,dx$

(b) $\displaystyle\int \frac{2x^2+x+16}{(x-2)(x^2+9)}\,dx$

$$\int \frac{x-1}{x(x+1)^2}\,dx$$

$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}=\frac{x-1}{x(x+1)^2}$$

$$A(x+1)^2+Bx(x+1)+Cx=x-1$$

$$A(x^2+2x+1)+Bx^2+Bx+Cx=x-1$$

$$Ax^2+2Ax+A+Bx^2+Bx+Cx=x-1$$

$$A+B=0 \qquad 2A+B+C=1 \qquad A=-1$$

$$-1+B=0 \qquad 2(-1)+1+C=1$$

$$B=1 \qquad C=2$$

$$-\int\frac{1}{x}\,dx+\int\frac{dx}{x+1}+2\int(x+1)^{-2}\,dx$$

$$-\ln x+\ln|x+1|-2(x+1)^{-1}+C$$

Practice Questions...

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