Warm Up

Evaluate each of the following:

$$\int \frac{1}{(16-x^2)^{\frac{3}{2}}}\,dx$$ (UNB: 2005)

$$\int \frac{\sqrt{2-x^2}}{x}\,dx$$ (UNB: 2004)

$$\int (\csc^2 t \cot^2 t)\; dt$$ (UNB: 2004)

$$\int \frac{\ln(\ln x)}{x}\,dx$$ (StFX: 2005)

Integration using Partial Fractions

Simplify: $\frac{3}{x-5}+\frac{2}{x+4}$

We want to reverse the process of
finding a common denominator...

Express as partial fractions: $\frac{5x+2}{x^2-x-20}$

1. Factor the denominator: $\frac{5x+2}{(x-5)(x+4)}$

2. Separate into partial fractions:

$$\frac{A}{x-5}+\frac{B}{x+4}=\frac{5x+2}{x^2-x-20}$$

3. Find common denominator and solve for A and B:

Now let's evaluate the following integral...

$$\int \frac{(5x+2)dx}{x^2 - x - 20}$$

Here is another example...

$$\int \frac{xdx}{x^2 - 3x + 2}$$

Some special situations involving partial fractions...

Note 1:

- If the degree of the numerator is the same as that of the denominator, or higher, we would have to take the preliminary step of first performing a long division.

Ex. $$\frac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1}$$

$$2x^2 + x + 1 \overline{)\, 2x^3 - 11x^2 - 2x + 2} = (x-6) + \frac{5x-4}{(x+1)(2x-1)}$$

Note 2:

- If the denominator has more than two linear factors, we must include a term corresponding to each factor.

Ex. $$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}$$

Note 3:

- If a linear factor is repeated, we need to include extra terms in the partial fraction expression.

Ex. $$\frac{x}{(x+3)^2(x-2)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-2}$$

Note 4:

- When we factor the denominator as far as possible, it may happen that we end up with an irreducible quadratic factor of the form $ax^2 + bx + c$, where the discriminant is negative. Then the corresponding partial fraction is of the form...

$$\frac{Ax+B}{ax^2+bx+c}$$

where A and B are constants to be determined. This term can be integrated by completing the square and by using the integration formula...

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$ ***(or a trig. substitution)***

Examples...

(a) $\displaystyle\int \frac{x-1}{x(x^2+2x+1)}\,dx$

(b) $\displaystyle\int \frac{2x^2+x+16}{(x-2)(x^2+9)}\,dx$

$x(x+1)^2$

$\frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$

(b) $\displaystyle\int \frac{2x^2+x+16}{(x-2)(x^2+9)}\,dx$

$$\frac{A}{x-2}+\frac{Bx+C}{x^2+9}=\frac{2x^2+x+16}{(x-2)(x^2+9)}$$

$$\frac{A(x^2+9)+(Bx+C)(x-2)}{(x-2)(x^2+9)}=\frac{2x^2+x+16}{(x-2)(x^2+9)}$$

$$Ax^2+9A+Bx^2-2Bx+Cx-2C=2x^2+x+16$$

$A+B=2$ $\quad (-2B+C=1)\times 2 \quad$ $9A-2C=16$

$A=2-B$ $\qquad\qquad\qquad\qquad$ $9(2-B)-2C=16$

$18-9B-2C=16$

$(-9B-2C=-2)\times -1$

$9B+2C=2$

$$\begin{aligned}-4B+2C&=2\\ 9B+2C&=2\\ \hline -13B&=0\end{aligned}$$

$\underline{B=0}$

$-2(0)+C=1$

$\underline{C=1}$

$A=2-0$

$\underline{A=2}$

$$2\int\frac{dx}{x-2}+\int\frac{dx}{x^2+9}$$

$$2\int\frac{dx}{x-2}+\int\frac{dx}{9\left(\frac{1}{9}x^2+1\right)}$$

$$2\int\frac{dx}{x-2}+\frac{(3)1}{9}\int\frac{\left(\frac{1}{3}\right)dx}{\left(\frac{1}{3}x\right)^2+1}$$

$$=2\ln|x-2|+\frac{1}{3}\tan^{-1}\left(\frac{1}{3}x\right)+C$$

Practice Questions...

Page 404 - 405

#9, 10, 11, 13, 14, 17, 19, 21, 23, 25, 27

Kansas State University

Let's borrow a few examples from final exams given to the Wildcats over the past years...

Evaluate $\int \frac{(x+2)(x+4)}{(x+1)(x+3)(x+5)}\,dx$.

$$\int \frac{x^3 + 2x - 1}{x^2 - 1}\,dx$$

Evaluate $\int \frac{dx}{x^2 + 6x + 13}$

Evaluate $\int_0^1 2^{-x}\,dx$.

$$\int \frac{\sqrt{x^2 - 1}}{x}\,dx$$

Evaluate $\int \frac{\cos^5(x)}{\sin^2(x)}\,dx$.

Evaluate $\int \frac{(x+2)(x+4)}{(x+1)(x+3)(x+5)}\,dx$.

$$\frac{A}{x+1}+\frac{B}{x+3}+\frac{C}{x+5}=\frac{x^2+6x+8}{\cdots}$$

$$A(x^2+8x+15)+B(x^2+6x+5)+C(x^2+4x+3)=x^2+6x+8$$

$A+B+C=1$ $\quad 8A+6B+4C=6 \quad 15A+5B+3C=8$

$A=1-B-C$

$8(1-B-C)+6B+4C=6 \qquad 15(1-B-C)+5B+3C=8$

$-2B-4C=-2 \qquad -10B-12C=-7$

$\times 3\ (2B+4C=2) \qquad 10B+12C=7$

$$\begin{aligned}6B+12C&=6\\ 10B+12C&=7\\ \hline -4B&=-1\\ B&=\tfrac{1}{4}\end{aligned}$$

$2\left(\frac{1}{4}\right)+4C=2$

$4C=\frac{3}{2}$

$C=\frac{3}{8}$

$A=1-\frac{1}{4}-\frac{3}{8}$

$A=\frac{8-2-3}{8}$

$A=\frac{3}{8}$

$$\frac{3}{8}\int\frac{dx}{x+1}+\frac{1}{4}\int\frac{dx}{x+3}+\frac{3}{8}\int\frac{dx}{x+5}$$

$$=\frac{3}{3}\ln|x+1|+\frac{1}{4}\ln|x+3|+\frac{3}{8}\ln|x+5|+C$$

$$\int \frac{x^3 + 2x - 1}{x^2 - 1}\, dx$$

$$\begin{array}{r|l} & x \\ x^2-1 & x^3+2x-1 \\ & x^3 - x \\ \hline & 3x - 1 \end{array}$$

$$\int x + \frac{3x-1}{x^2-1}\, dx$$

$$\int x\, dx + \int \frac{3x-1}{x^2-1}\, dx$$

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{3x-1}{x^2-1}$$

$$A(x+1) + B(x-1) = 3x-1$$

$$A + B = 3 \qquad A - B = -1$$

$$\begin{array}{r} A - B = -1 \\ A + B = 3 \\ \hline 2A = 2 \end{array}$$

$$A = 1 \qquad 1 + B = 3$$

$$B = 2$$

$$\int x\, dx + \int \frac{dx}{x-1} + 2\int \frac{dx}{x+1}$$

$$= \frac{x^2}{2} + \ln|x-1| + 2\ln|x+1| + C$$

Using Integration Tables...

Here is an example of integration tables

How do these things work???

#29 $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}x\sqrt{x^2 \pm a^2} \pm \frac{1}{2}a^2 \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

ex. $\int \sqrt{x^2 + 7}\,dx$

$a = \sqrt{7}$

$= \frac{1}{2}x\sqrt{x^2+7} + \frac{7}{2}\ln\left|x + \sqrt{x^2+7}\right| + C$

$$\int e^{6x} \cos \pi x \, dx$$

Integration Tables.pdf