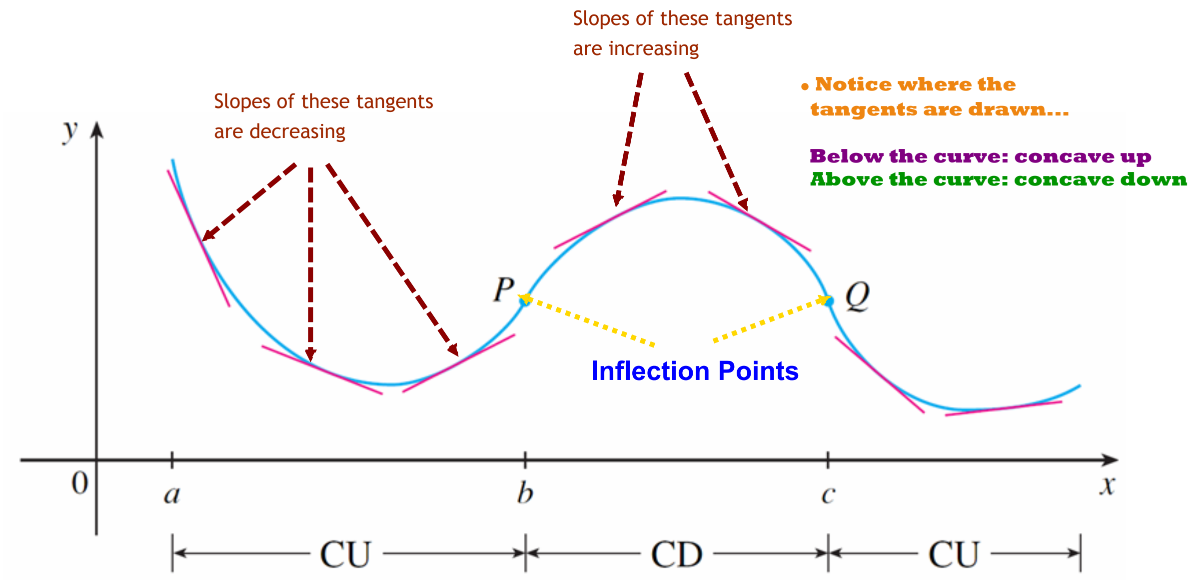
Concavity

A function (or its graph) is called **concave upward** on an interval I if f' is an increasing function on I. It is called **concave downward** on I if f' is decreasing on I.



- A point where a curve changes its direction of concavity is called an ***inflection point***.

If $f'(x) > 0$ then $f(x)$ is increasing,
so if $f''(x) > 0$ then $f'(x)$ is increasing.

Concavity Test Positive Up

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Negative Down

Thus there is a point of inflection at any point where the second derivative changes sign.

Second Derivative Test for Local Extrema

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

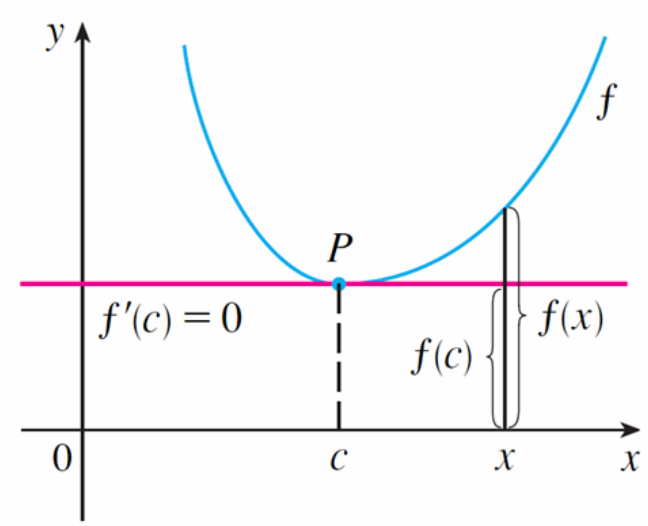


FIGURE 6

$f''(c) > 0$, f is concave upward

at $x = b$

$f'(b) = 0$

$f''(b) = -7$

$\therefore$ b is a local max.

Example:

Examine the function $f(x) = x^4 - 4x^3$ with respect to…

- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

$f'(x) = 4x^3 - 12x^2$

$0 = 4x^2(x-3)$

$x = 0, 3$

	$4x^2$	$x-3$	f'	f
$(-\infty, 0)$	+	−	−	Dec
$(0, 3)$	+	−	−	Dec
$(3, \infty)$	+	+	+	Inc

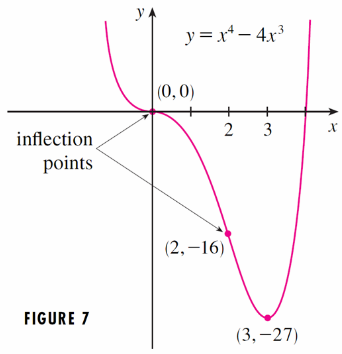


FIGURE 7

Local Max

None

Local Min.

$(3, -27)$

Concavity

$f''(x) = 12x^2 - 24x$

$0 = 12x(x-2)$

$x = 0, 2$

	$12x$	$x-2$	f''	f
$(-\infty, 0)$	−	−	+	Up
$(0, 2)$	+	−	−	Down
$(2, \infty)$	+	+	+	Up

Inflection Points

$(0, 0)$ & $(2, -16)$

x-Intercepts

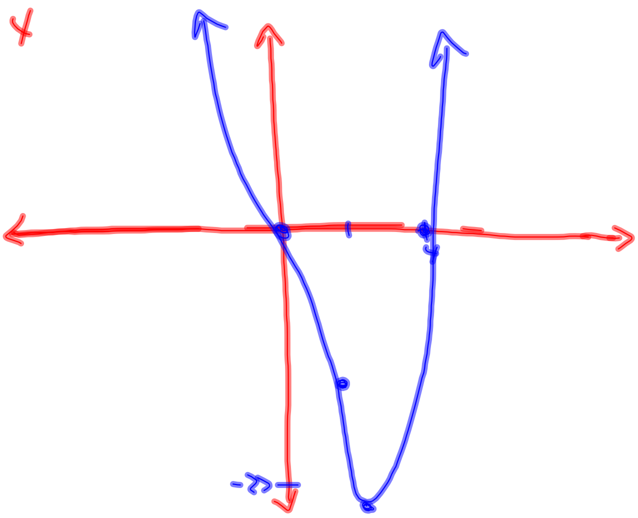
$x^4 - 4x^3 = 0$

$x^3(x-4) = 0$

$x = 0, 4$

y-Intercepts

$y = 0$



Symmetry -

Even Symmetry

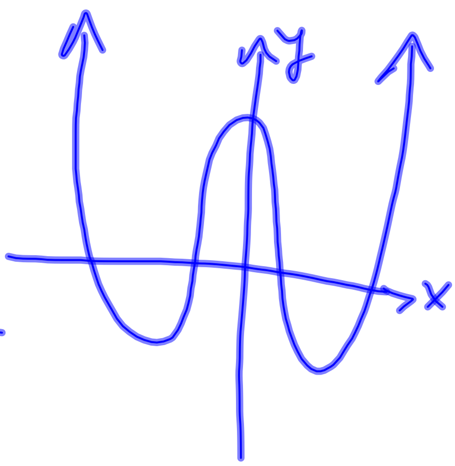
$f(x) = f(-x)$

symmetric about y-axis (line)

$y = x^4 + 2x^2 + 6$

$f(-x) = (-x)^4 + 2(-x)^2 + 6$

$f(-x) = x^4 + 2x^2 + 6$



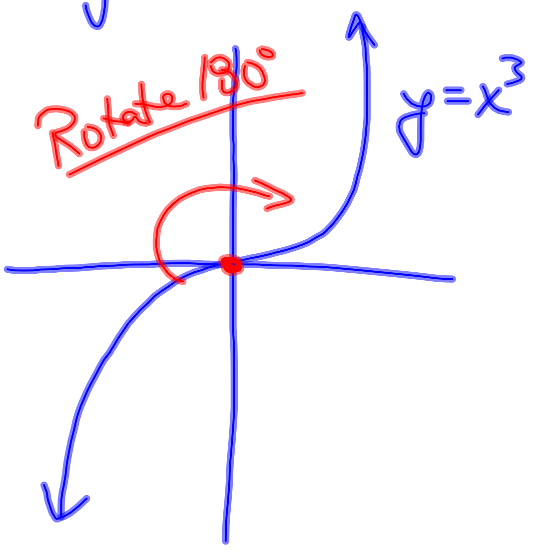
ODD Symmetry

$f(-x) = -f(x)$

$f(x) = x^3 + 1$

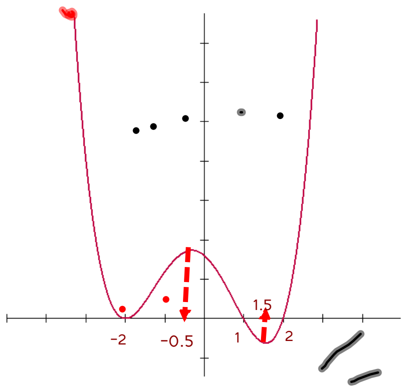
$f(-x) = (-x)^3 = -x^3$

Symmetric about the origin (Point)



Warm-Up

The graph of the derivative of a function f on the interval [-4 , 4] is shown below:



(a) On what intervals is f increasing?

$[-4, 1)$ & $(2, 4]$

$f'(x)$ Positive

(b) On what intervals is the graph of f concave up?

$(-2, -0.5)$ & $(1.5, 4)$

(c) At what x-coordinate does f have local extrema?

1 & 2

(d) What are the x-coordinates of all inflection points of the graph of f?

$x = -2, -0.5, 1.5$

Determine the regions of concavity and all inflection points for the following function: $f(x) = x^4 - 2x^3 - 12x^2 + 3$

$f'(x) = 4x^3 - 6x^2 - 24x$

① $f''(x) = 12x^2 - 12x - 24$

② Critical Values of $f''(x)$:

$0 = 12(x^2 - x - 2)$

$0 = 12(x-2)(x+1)$

$x = 2, -1$

③ 2nd derivative sign table:

	12	x − 2	x + 1	f''	f
$(-\infty, -1)$	+	−	−	+	Up
$(-1, 2)$	+	−	+	−	Down
$(2, \infty)$	+	+	+	+	Up

Inflection Points:

$(-1, -6)$ & $(2, -45)$

Example:

Using the function: $f(x) = \dfrac{x^2}{x-7}$

Determine each of the following...

- Intercepts
- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

$$f'(x) = \frac{2x(x-7) - x^2}{(x-7)^2}$$

$$f'(x) = \frac{2x^2 - 14x - x^2}{(x-7)^2}$$

$$f'(x) = \frac{x^2 - 14x}{(x-7)^2}$$

$$\boxed{f'(x) = \frac{x(x-14)}{(x-7)^2}}$$

$f''(x) =$

$$f''(x) = \frac{(2x-14)(x-7)^2 - (x^2-14x)(2(x-7))}{(x-7)^4}$$

$$f''(x) = \frac{2\cancel{(x-7)}\left[(x-7)^2 - x^2 + 14x\right]}{(x-7)^{\cancel{4}}}$$

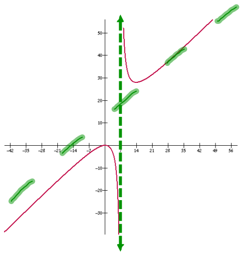
$$f''(x) = \frac{2(\cancel{x^2} \cancel{-14x} + 49 \cancel{-x^2} \cancel{+14x})}{(x-7)^3}$$

$$\boxed{f''(x) = \frac{98}{(x-7)^3}}$$

Let's look at homework question…

Example:

Using the function: $f(x) = \dfrac{x^2}{x-7} = \dfrac{14^2}{7}$



Determine each of the following…

- Intercepts
- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

$f'(x) = \dfrac{x(x-14)}{(x-7)^2}$ $f''(x) = \dfrac{98}{(x-7)^3}$

Intercepts:

x-Int. (y=0)

$0 = \dfrac{x^2}{(x-7)}$

$x^2 = 0$

$x = 0$

y-Int. (x=0)

$y = \dfrac{0^2}{0-7}$

$y = 0$

$(0,0)$

Max/Min.

$f'(x) = \dfrac{x(x-14)}{(x-7)^2}$

Critical Values

$x = 0, 14, 7$

	x	$x-14$	$(x-7)^2$	f'	f
$(-\infty,0)$	−	−	+	+	Inc
$(0,7)$	+	−	+	−	Dec
$(7,14)$	+	−	+	−	Dec
$(14,\infty)$	+	+	+	+	Inc

Local Max.

$(0,0)$

Local Min.

$(14,28)$

Concavity

$f''(x) = \dfrac{98}{(x-7)^3}$

C. Value => $x = 7$

	98	$(x-7)^3$	f''	f
$(-\infty,7)$	+	−	−	Down
$(7,\infty)$	+	+	+	Up

Inflection Point: None

$(7, \ )$ undefined

Asymptotes:

Horizontal:

$F(x) = \dfrac{x^2}{x-7}$

$\lim_{x\to\pm\infty} \dfrac{\frac{x^2}{x^2}}{\frac{x}{x^2} - \frac{7}{x^2}}$

$= \dfrac{1}{0-0}$

None

Vertical: (Set Den. = 0)

$x - 7 = 0$

$x = 7$

$\lim_{x\to 7^-} \dfrac{x^2}{x-7} = \dfrac{49}{\text{small}(-)} \to -\infty$

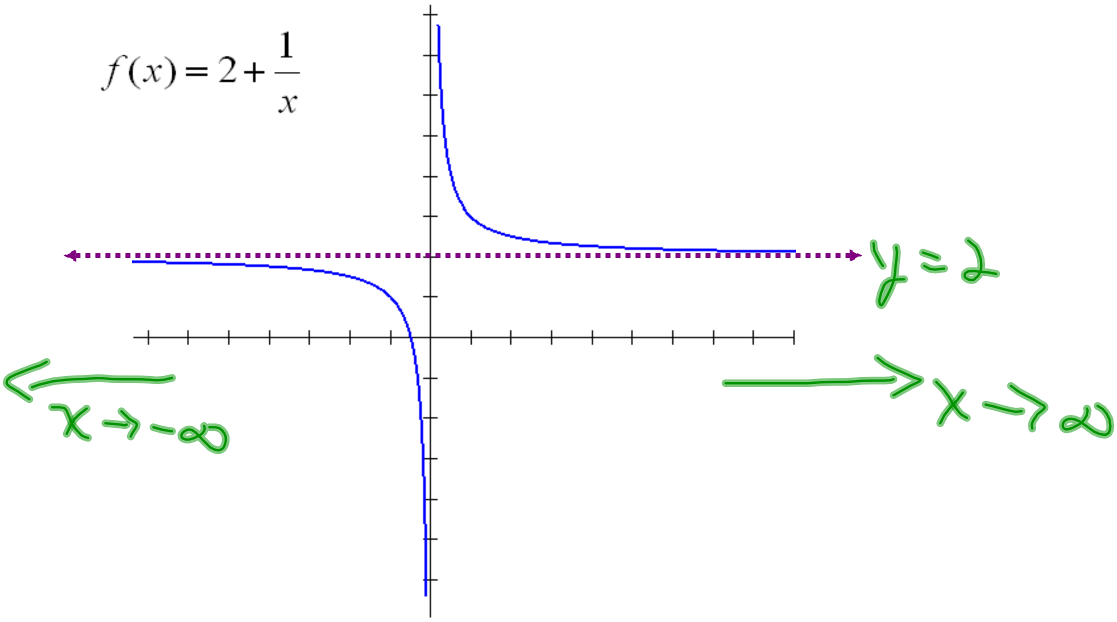
$\lim_{x\to 7^+} \dfrac{x^2}{x-7} \quad \dfrac{49}{\text{small}(+)} \to \infty$

Asymptotes

Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b$$



Examine the limits of $f(x)$ as x approaches $\pm\infty$

$$\lim_{x\to\pm\infty}\left(2 + \frac{1}{x}\right)$$

$$= 2 + 0 \qquad \frac{1}{\pm\infty} = 0$$

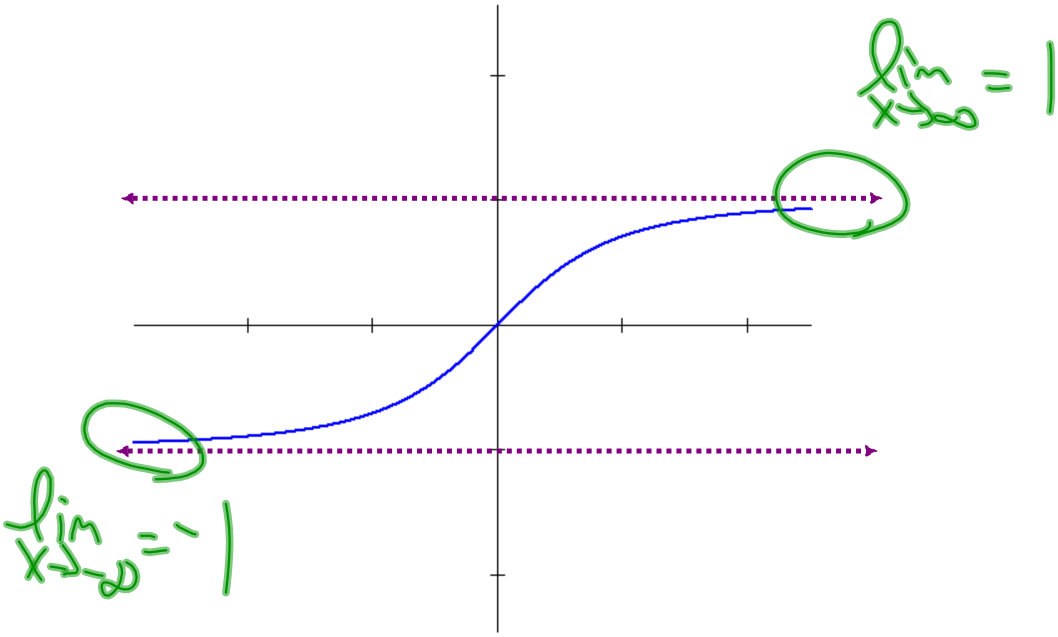
$\therefore$ $y = 2$ is a horizontal asymptote

* $$\lim_{x\to\infty} \frac{x^2 - x}{3x^2 - 2}$$??

Divide by highest Power!!

There can be more than one horizontal asymptote.

Examine the function $f(x) = \dfrac{x}{\sqrt{x^2+1}}$



Examine the limits of $f(x)$ as x approaches $\pm\infty$

Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

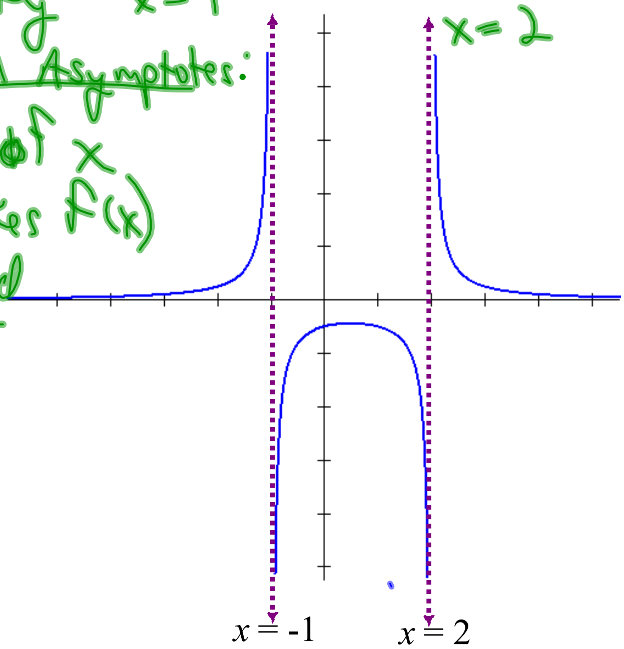
$$\lim_{x \to a^-} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = \pm\infty$$

Example: $f(x) = \dfrac{1}{x^2 - x - 2}$

① Identify Vertical Asymptotes:

Any value of x that makes $f(x)$ undefined

$x = -1$ $\quad$ $x = 2$



Set Den. = 0

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2, -1$

Use limits to examine the behaviour of the function near the asymptotes

Describe Behaviour:

$$\lim_{x \to -1^-} \frac{1}{(x+1)(x-2)} = \frac{1}{(0^-)(-)} = \frac{1}{0^+} \to \infty$$

$$\lim_{x \to -1^+} \frac{1}{(0^+)(-)} = \frac{1}{0^-} \to -\infty$$

Sketch the following function:

$$f(x)=\frac{8(x-2)}{x^2} \qquad f'(x)=\frac{-8(x-4)}{x^3} \qquad f''(x)=\frac{16(x-6)}{x^4}$$

Be sure to examine...

- Intercepts
- Asymptotes (*vertical and horizontal)*
- Regions of increase/decrease
- Local extrema
- Regions where concave up/down
- Inflection points

Intercepts

x-Int.

$0=\frac{8(x-2)}{x^2}$

$x=2$

$(2,0)$

y-Int.

$y=\frac{8(0-2)}{0^2}$

undefined

$\therefore$ None

Asymptotes

Horizontal

$\lim_{x\to\pm\infty}\frac{\frac{8x}{x}-\frac{16}{x}}{\frac{x^2}{x^2}}=\frac{0-0}{1}$

$\boxed{y=0}$

Vertical

$x^2=0$

$x=0$

$\lim_{x\to 0^-}\frac{8(x-2)}{x^2}\to-\infty$

$\lim_{x\to 0^+}\frac{8(x-2)}{x^2}\to-\infty$

Inc/Dec.

$f'(x)=\frac{-8(x-4)}{x^3}$

Critical Values:

$x=4,0$

	-8	$x-4$	x^3	f'	f
$(-\infty,0)$	$-$	$-$	$-$	$-$	Dec
$(0,4)$	$-$	$-$	$+$	$+$	Inc
$(4,\infty)$	$-$	$+$	$+$	$-$	Dec

Local Max.

$(4,1)$

Local Min.

None

Concavity

$0=\frac{16(x-6)}{x^4}$

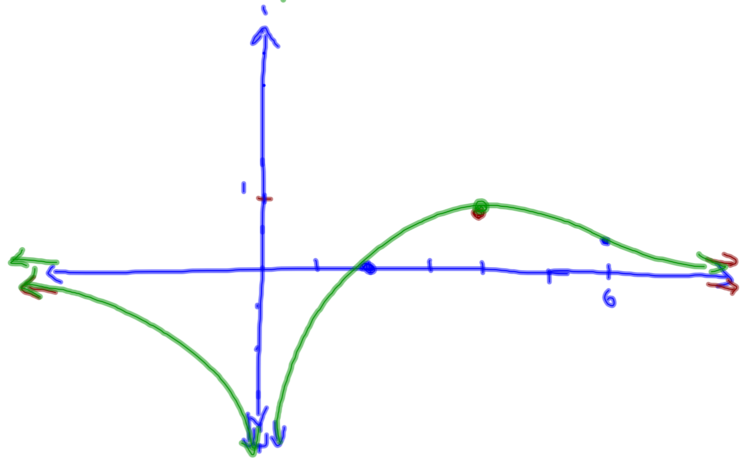
Critical Values

$x=6,0$

	16	$x-6$	x^4	f''	f
$(-\infty,0)$	$+$	$-$	$+$	$-$	Down
$(0,6)$	$+$	$-$	$+$	$-$	Down
$(6,\infty)$	$+$	$+$	$+$	$+$	Up

Inflection Point(s)

$(6,\frac{8}{9})$



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