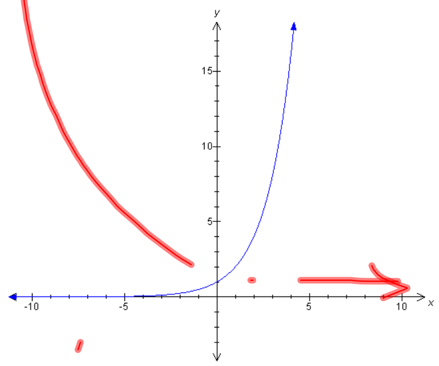
Differentiating Exponential Functions

What is an exponential function?

$y=\left(\frac{1}{2}\right)^x$

$y = 2^x$ Base must be a Real number



When you do not have a rule to differentiate resort to the definition...

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Let's try and differentiate $y = a^x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

This factor does not depend on h, therefore we can move to the front of the limit

Thus we now have...

$$f'(x) = a^x \lim_{h\to 0}\frac{a^h-1}{h}$$

What would be the value of $f'(0)$?

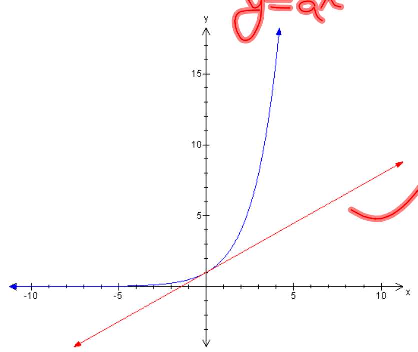
$a^0=1$

$$\lim_{h\to 0}\frac{a^h-1}{h} = f'(0)$$

What would this represent in terms of slope??

slope of this tangent

$y=a^x$



We have determined that $f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$

$F(x) = a^x$

Same thing!

and that $\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$

$f(x) = 7^x$

$f'(x) = 7^x \cdot$

Slope of tangent to $y = 7^x$ at $x = 0$

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "*e*"...Euler's Number

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e^(1)
      2.718281828
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...

This leads to the following definition...

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$y = e^x$

What does this mean geometrically?

- Geometrically, this means that
  - of all the exponential functions $y = a^x$,
  - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 .

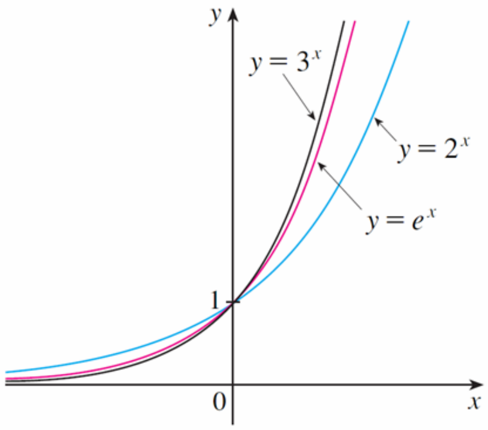

FIGURE 6

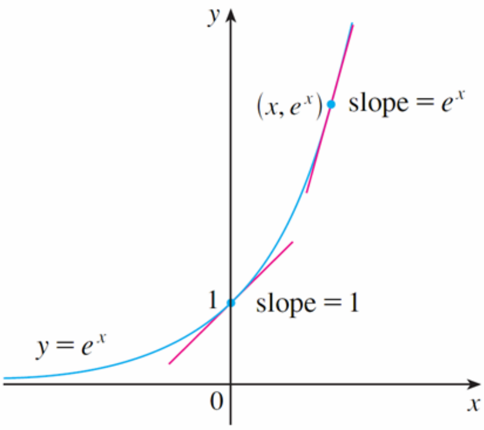

FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

This is the ONLY function that is its own derivative $\dashrightarrow$ $f(x) = e^x$, $f'(x) = e^x$

In General...

$$\boxed{\frac{d(e^u)}{dx} = e^u \bullet du}$$

ex.

① $y = e^{x^2}$

$y' = e^{x^2} \cdot (2x)$

② $y = e^{3+\cos x^2}$

$y' = e^{3+\cos x^2}(-\sin x^2(2x))$

③ $y = \dfrac{\csc^{-1} e^{x^5} - \cot^3 e^{5x}}{e^{\tan^{-1}\sqrt{x}}}$ $(\cot e^{5x})^3$

$$y' = \frac{\left[\dfrac{-e^{x^5}(5x^4)}{e^{x^5}\sqrt{e^{2x^5}-1}} - 3(\cot e^{5x})^2(-\csc^2 e^{5x}(e^{5x})(5))\right] \cdot e^{\tan^{-1}\sqrt{x}} - (\csc^{-1} e^{x^5} - \cot e^{5x})\left[e^{\tan^{-1}\sqrt{x}}\left(\dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x}\right)\right]}{\left(e^{\tan^{-1}\sqrt{x}}\right)^2}$$

3/ $e^{x^2y^3} = 3x^2 - 5xy$

$$e^{x^2y^3}\left(2xy^3 + x^2(3y^2)\frac{dy}{dx}\right) = 6x - \left(5y + 5x\frac{dy}{dx}\right)$$

$$2xy^3e^{x^2y^3} + 3x^2y^2e^{x^2y^3}\frac{dy}{dx} = 6x - 5y - 5x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 5y - 2xy^3e^{x^2y^3}}{3x^2y^2e^{x^2y^3} + 5x}$$

Practice Exercises

Page 367

#4, 5, 6, 8, 9, 10,