TEST : LIMITS

ADVANCED MATHEMATICS WITH CALCULUS 120

NAME: Solutions

MARCH 2012

1. Evaluate each of the following limits, indicating if they do not exist. Clearly show all work! [32]

(a) $\lim_{x\to 3}\frac{x^3-27}{x^4-81}$

$\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x^2-9)(x^2+9)}$

$\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)(x^2+9)}$

$=\frac{27}{(6)(18)}$

$=\boxed{\frac{1}{4}}$

(b) $\lim_{x\to 0}\frac{\sin 5x^2}{7x^2}$

$\lim_{x\to 0}\frac{5}{7}\left(\frac{\sin 5x^2}{5x^2}\right)$

$=\boxed{\frac{5}{7}}$

(c) $\lim_{w\to\infty}\frac{(2-3w)(1-2w^3)}{(3w^2+1)^2}$

$\lim_{w\to\infty}\frac{\frac{2}{w^4}-\frac{4w^3}{w^4}-\frac{3w}{w^4}+\frac{6w^4}{w^4}}{\frac{9w^4}{w^4}+\frac{6w^2}{w^4}+\frac{1}{w^4}}$

$=\frac{6}{9}$

$=\boxed{\frac{2}{3}}$

(d) $\lim_{x\to -7^+}\frac{|x+7|}{x^2-49}$ ← $(x+7)(x-7)$

$\lim_{x\to -7^+}\frac{|-6.999\ldots+7|}{(-6.999\ldots+7)(-14)}$

$=\boxed{\frac{-1}{14}}$

(e) $\lim_{x\to 0}\frac{\frac{1}{x+2}-\frac{1}{2}}{3x}$

$\lim_{x\to 0}\left[\frac{2-(x+2)}{2(x+2)}\right]\cdot\frac{1}{3x}$

$\lim_{x\to 0}\frac{-\cancel{x}}{2(x+2)(3\cancel{x})}$

$=\frac{-1}{2(2)(3)}$

$=\boxed{\frac{-1}{12}}$

(f) $\lim_{x\to 0}\frac{5x^3+x^4}{\sin^3 3x}$

$\frac{x^3(5+x)}{\sin^3 3x}$

$\lim_{x\to 0}\left(\frac{3x}{\sin 3x}\right)^3\frac{5+x}{27}$

$=(1)^3\left(\frac{5}{27}\right)$

$=\boxed{\frac{5}{27}}$

(g) $\lim_{x\to 4}\frac{\sqrt{x+5}-\sqrt{3x-3}}{x^2-4x}\left(\frac{\sqrt{x+5}+\sqrt{3x-3}}{\sqrt{x+5}+\sqrt{3x-3}}\right)$

$\lim_{x\to 4}\frac{(x+5)-(3x-3)}{x(x-4)(\sqrt{x+5}+\sqrt{3x-3})}$

$\lim_{x\to 4}\frac{8-2x}{x(x-4)(\sqrt{x+5}+\sqrt{3x-3})}$

$\lim_{x\to 4}\frac{2\cancel{(4-x)}^{-1}}{x\cancel{(x-4)}(\sqrt{x+5}+\sqrt{3x-3})}$

$=\frac{-2}{4(3+3)}$

$=\boxed{\frac{-1}{12}}$

(h) $\lim_{x\to w}\frac{(x+5w)^2-36w^2}{x-w}$

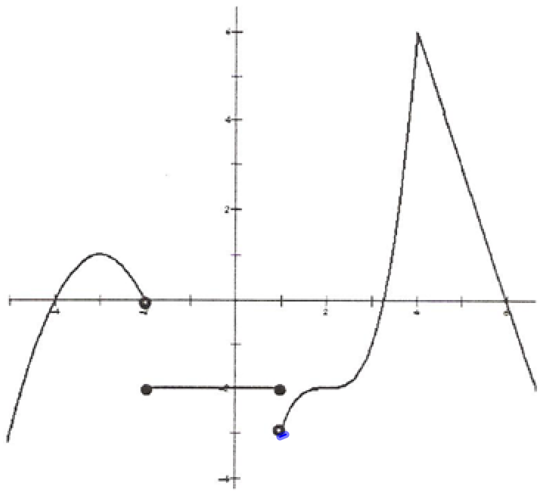
$\lim_{x\to w}\frac{(x+5w-6w)(x+5w+6w)}{x-w}$

$\lim_{x\to w}\frac{\cancel{(x-w)}(x+11w)}{\cancel{x-w}}$

$= w+11w$

$\boxed{=12w}$

2. Use the graph provided to fill in the blanks below. [6]



(a) $f(-2)=$ $\underline{-2}$

(b) $\lim_{x\to -2^-} f(x)=$ $\underline{0}$

(c) $\lim_{x\to 1^+} f(x)=$ $\underline{-3}$

(d) $f(4)=$ $\underline{6}$

(e) $\lim_{x\to 1} f(x)=$ $\underline{\text{DNE}}$

(f) $\lim_{x\to -2^+} f(x)=$ $\underline{-2}$

3. Use the **definition of a derivative** to determine $\frac{dy}{dx}$ given that $f(x)=\sqrt{3x+1}$. [5]

$f(x+h)=\sqrt{3(x+h)+1}$
$=\sqrt{3x+3h+1}$

$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\lim_{h\to 0}\frac{\sqrt{3x+3h+1}-\sqrt{3x+1}}{h}\left(\frac{\sqrt{3x+3h+1}+\sqrt{3x+1}}{\sqrt{3x+3h+1}+\sqrt{3x+1}}\right)$

$\lim_{h\to 0}\frac{(3x+3h+1)-(3x+1)}{h(\sqrt{3x+3h+1}+\sqrt{3x+1})}$

$\lim_{h\to 0}\frac{3\cancel{h}}{\cancel{h}(\sqrt{3x+3h+1}+\sqrt{3x+1})}$

$=\boxed{\frac{3}{2\sqrt{3x+1}}}$

4. Let $f(x)=\begin{cases} x-1, & if\ x<-1 \\ x^2-2, & if\ -1\le x<2 \\ 1, & if\ x=2 \\ -(x-1)^2+3, & if\ x>2 \end{cases}$

(a) Check $f(x)$ for any points of discontinuity. Clearly show your work for all continuity checks. Provide a mathematical reason to validate any point(s) where $f(x)$ is discontinuous. [6]

$x=-1$

$f(-1)=(-1)^2-2$
$=-1$

$\lim_{x\to -1^-} f(x) = -1-1 = -2$

$\lim_{x\to -1^+} f(x) = -1$

$\lim_{x\to -1} f(x)$ does not exist

$\therefore$ Discontinuous

$x=2$

$f(2)=1$

$\lim_{x\to 2^-} f(x) = (2)^2-2 = 2$

$\lim_{x\to 2^+} f(x) = -(2-1)^2+3 = 2$

$\lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ discontinuous

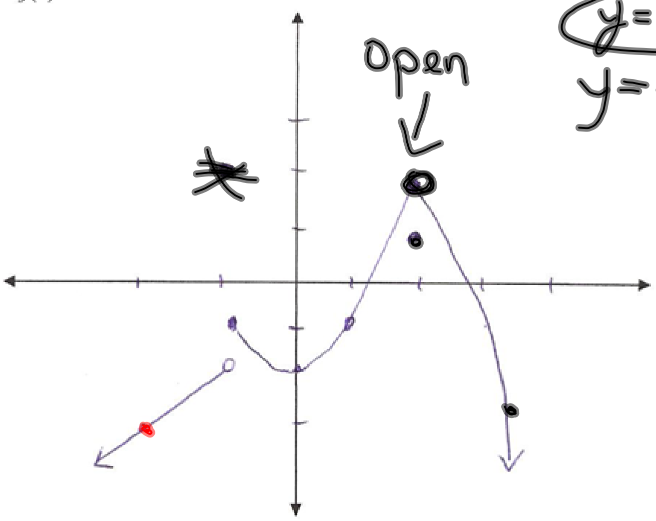
(b) Sketch $f(x)$. [4]

① $y=x-1$

x	y
-1	-2
-2	-3

② $y=x^2-2$
V(0,-2)

x	y
-1	-1
+2	2



Open

$y=a(x-h)^2+k$
$y=-(x-1)^2+3$
V(1,3)

x	y
2	2
3	-3

5. Determine the coordinates of any point(s) on the function $f(x)=x^2-3x+5$ where a tangent line would be parallel to the line $x+y=4$. **(Must use limits!)** [6]

same slope

$y=-x+4$
$m=-1$

$f(x+h)=(x+h)^2-3(x+h)+5$
$=x^2+2xh+h^2-3x-3h+5$

$y=-x+4$
$m=-1$
slope $=f'(x)$

$\lim_{h\to 0}\dfrac{(\cancel{x^2}+2xh+h^2-\cancel{3x}-3h+\cancel{5})-(\cancel{x^2}-\cancel{3x}+\cancel{5})}{h}$

$\lim_{h\to 0}\dfrac{h(2x+h-3)}{h}$

$f'(x)=2x-3$

$2x-3=-1$
$2x=2$
$x=1$
$f(1)=1-3+5$
$=3$

$\boxed{(1,3)}$

6. Determine the equation of the ***normal*** to the curve $y = \frac{2-x}{x+1}$ at $x = -2$. **(Must use limits!)** [6]

$\lim_{h \to 0} \frac{\frac{(2-x-h)}{x+h+1} - \frac{2-x}{x+1}}{h}$

$f(x+h) = \frac{2-x-h}{x+h+1}$

$\lim_{h \to 0} \frac{(2-x-h)(x+1) - (2-x)(x+h+1)}{(x+h+1)(x+1)(h)}$

at $x = -2$ ... $f(-2) = \frac{2-(-2)}{-2+1}$

$= \frac{4}{-1}$

$\lim_{h \to 0} \frac{2x+2-x^2-x-xh-h-2x-2h-2+x^2+xh+x}{(x+h+1)(x+1)(h)}$

$(-2,-4)$

$\lim_{h \to 0} \frac{-3\cancel{h}}{(x+h+1)(x+1)\cancel{h}}$

$y + 4 = \frac{1}{3}(x+2)$

$f'(x) = \frac{-3}{(x+1)^2} \Rightarrow$ at $x = -2$

slope $= \frac{-3}{(-2+1)^4}$

$= -3$

$\therefore m = \frac{1}{3}$

$3y + 12 = x + 2$

$\boxed{x - 3y - 10 = 0}$

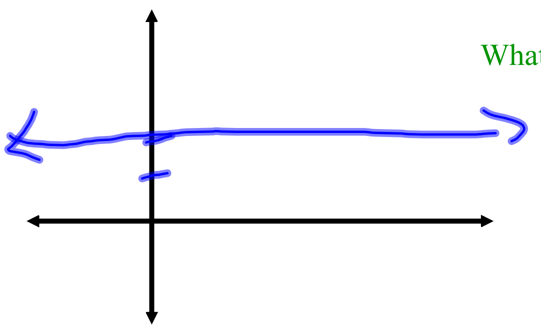
$y = \frac{x}{3} - \frac{10}{3}$

BONUS
Find a parabola with equation $y = ax^2 + bx + c$ that has slope -4 at $x = 1$, slope 8 at $x = -1$, and passing through the point (1,6). [4]

Differentiation Rules

I. Constant Functions

- Sketch the function $y = 2$



What is the slope?

$m=0$

$y=2$

The derivative of a constant will always be equal to "0".

Formal Proof:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0 = 0$$

II. Power Functions

We want to come up with a rule to differentiate functions of the form $f(x) = x^n$, $x \in R$

Here are a couple derivatives that we would have already looked at using limits:

$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

Using the definition of a derivative to differentiate $f(x) = x^4$ would lead to ...

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

$f(x) = x^{15}$

$f'(x) = 15x^{14}$

Do you see a pattern emerging?

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Let's practice using the power rule...

Differentiate each of the following functions:

1. $f(x) = x^{25}$

$f'(x) = 25x^{24}$

2. $f(x) = x^{-5}$

$f'(x) = -5x^{-6}$

$\sqrt[M]{x^{\omega}} = x^{\frac{\omega}{M}}$

$\frac{7}{3} - 1$

$\frac{7}{3} - \frac{3}{3}$

3. $f(x) = \dfrac{1}{x^{10}}$

$f(x) = x^{-10}$

$f'(x) = -10x^{-11}$

4. $f(x) = \sqrt[3]{x^7}$

$f(x) = x^{7/3}$

$f'(x) = \frac{7}{3}x^{4/3}$

Constant Multiples

- The following formula says that the derivative of a constant multiplied by a function is the constant multiplied by the derivative of the function:

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

EXAMPLE 4

(a) $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$

(b) $\frac{d}{dx}(-x) = \frac{d}{dx}[(-1)x] = (-1)\frac{d}{dx}(x) = -1(1) = -1$

$y = 10x^7$
$y' = 70x^6$

$y = 3x^4$
$y' = 3(4x^3)$
$y' = 12x^3$

Sums and Differences

- These next rules say that the derivative of a sum (difference) of functions is the sum (difference) of the derivatives:

The Sum Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

The Difference Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

$f(x) = 3x^8 + x^7 - 6x^{-2}$
$f'(x) = 24x^7 + 7x^6 + 12x^{-3}$

Demonstrate what this all means...

Differentiate each of the following:

$2x^{1/2}$ $5x^{-1}$ constant $\frac{1}{2}-\frac{2}{2}$

1. $f(x)=2x^{5}-3x^{-3}+2\sqrt{x}-\frac{5}{x}+\pi^{2}$

$f'(x)=10x^{4}+9x^{-4}+1x^{-\frac{1}{2}}+5x^{-2}+0$

2. $f(x)=\left(2x^{3}-5\right)^{2}$

$f(x)=4x^{6}-20x^{3}+25$

$f'(x)=24x^{5}-60x^{2}$

$\frac{x^{3}}{x^{1/4}}=\frac{x^{\frac{12}{4}}}{x^{1/4}}$

3. $f(x)=\frac{7\sqrt{x}}{4}-\frac{6}{x^{5}}+\frac{10x^{3}}{\sqrt[4]{x}}+6x^{30}-ex+9$

$f(x)=\frac{7}{4}x^{1/2}-6x^{-5}+10x^{11/4}+6x^{30}-ex+9$

$f'(x)=\frac{7}{8}x^{-1/2}+30x^{-6}+\frac{55}{2}x^{7/4}+180x^{29}-e$

Homework

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Attachments

Worksheet - Nature of the Roots.doc