Warm Up

Quiz: Derivative Rules

Implicit Differentiation

$f(x) = x^2 + 3$

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x.
- Examples
  - $x^2 + y^2 = 25$
  - $x^3 + y^3 = 6xy$

- The first equation could easily be rearranged for $y = \ldots$

$$y = \pm\sqrt{25 - x^2}$$ ← Actually gives two functions

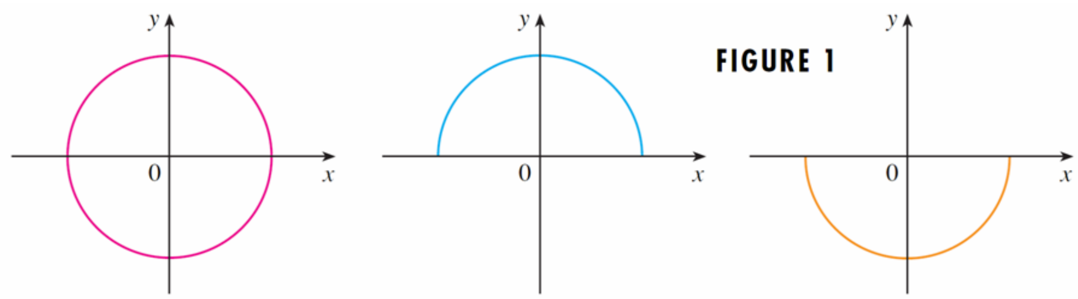

FIGURE 1

(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

- The other sample equation

$$x^3 + y^3 = 6xy$$

  - can be solved for y but
  - the results are very complicated.

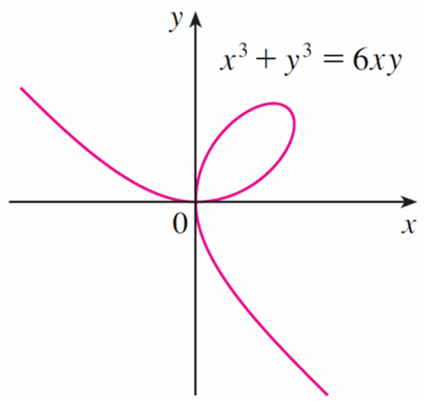

FIGURE 2 The folium of Descartes

$y = x^3 \Longrightarrow y = (x)^3$

$y' = 3x^2$

$\frac{dy}{dx} = 3(x)^2(1)$

$\frac{dy}{dx}$

"derivative of y with respect to variable x"

$\frac{dy}{dx}$

$y = 3x^2$

$\frac{dy}{dx} = 6x$

$(x)^2 + (y)^2 = 25$

$2(x)' + 2(y)' \frac{dy}{dx} = 0$ Okay $2x + 2yy'$

$\frac{2y \frac{dy}{dx}}{2y} = \frac{-2x}{2y}$

$\frac{dy}{dx} = -\frac{x}{y}$ ← Need (x, y) Not Just "y"!!!

Implicit Differentiation

- There is a way called *implicit differentiation* to find *dy/dx* without solving for *y* :
  - First differentiate both sides of the equation with respect to *x* ;
  - Then solve the resulting equation for y' .
- We will always assume that the given equation does indeed define *y* as a differentiable function of *x* .

Example

- For the circle $x^2 + y^2 = 25$, find
  a) *dy/dx*
  b) an equation of the tangent at the point (3, 4) .

Solution:

Start by differentiating both sides of the equation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that *y* is a function of *x* and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus… $2x + 2y\dfrac{dy}{dx} = 0$

Solving for $\frac{dy}{dx}$ … $\dfrac{dy}{dx} = -\dfrac{x}{y}$

Therefore at the point (3,4) the equation of the tangent would be…

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate *f*:

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

Solution (cont'd)

- So $f'(3) = -\frac{3}{\sqrt{25-3^2}} = -\frac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.

- Note that although this problem could be done both ways, implicit differentiation was easier!

Example

- For the folium of Descartes $x^3 + y^3 = 6xy$,
  - Find y'
  - Find the tangent to the curve at the point (3, 3)
  - At what points on the curve is the tangent line horizontal?

$$x^3 + y^3 = 6xy$$

Requires Product Rule

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{\frac{dy}{dx}(3y^2 - 6x)}{(3y^2 - 6x)} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Example:

Given $x^2 - 3x^3y^2 + y^2 = 5xy^3 - 4$

Find $\frac{dy}{dx}$

$3x^3 2y \rightarrow 6x^3y$

$15xy^2\frac{dy}{dx}$

$$2x - \left[9x^2y^2 + 3x^3\left(2y\frac{dy}{dx}\right)\right] + 2y\frac{dy}{dx} = 5\left(y^3 + x\left(3y^2\frac{dy}{dx}\right)\right)$$

$$\frac{dy}{dx}\left(-6x^3y + 2y - 15xy^2\right) = 5y^3 - 2x + 9x^2y^2$$

$$\frac{dy}{dx} = \frac{5y^3 - 2x + 9x^2y^2}{-6x^3y + 2y - 15xy^2}$$

$$7x^5y^8 + 8y^7 - x^3 = 24x^2y^3$$

$$35x^4y^8 + 7x^5(8y^7)\frac{dy}{dx} + 56y^6\frac{dy}{dx} - 3x^2 = 48xy^3 + 24x^2(3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx}(56x^5y^7 + 56y^6 - 72x^2y^2) = 48xy^3 - 35x^4y^8 + 3x^2$$

$$\frac{dy}{dx} = \frac{48xy^3 - 35x^4y^8 + 3x^2}{56x^5y^7 + 56y^6 - 72x^2y^2}$$

Example:

Find $\frac{dy}{dx}$, given the curve $x^2 - 3xy = (5x^2 - 8y)^5$.

$$2x - 3y - 3x\frac{dy}{dx} = 5(5x^2 - 8y)^4\left(10x - 8\frac{dy}{dx}\right)$$

☺$\left(10x - 8\frac{dy}{dx}\right)$

☺$(10x) -$ ☺ $8\frac{dy}{dx}$

$$2x - 3y - 3x\frac{dy}{dx} = 50x(5x^2 - 8y)^4 - 40(5x^2 - 8y)^4\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(40(5x^2 - 8y)^4 - 3x\right) = 50x(5x^2 - 8y)^4 - 2x + 3y$$

$$\frac{dy}{dx} = \frac{50x(5x^2 - 8y)^4 - 2x + 3y}{40(5x^2 - 8y)^4 - 3x}$$

Homework

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1 d, f, h

2 c, d

3 c, d

5 a

6 a, b, c

Attachments

Worksheet - Average Rate of Change #2.doc

Worksheet - Instantaneous Rate of Change.doc