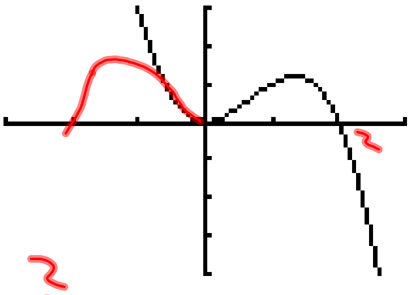
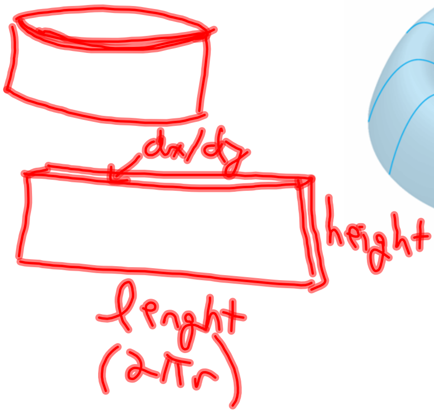
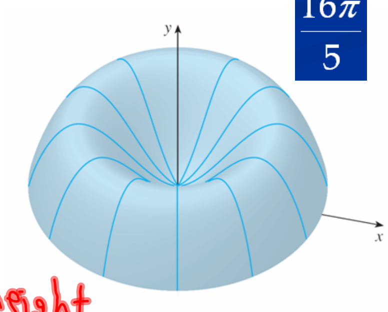
Warm Up

Determine the volume of the solid obtained by rotating the region enclosed by the curve $f(x) = 2x^2 - x^3$ and the x-axis about the y-axis.





$\frac{16\pi}{5}$



$$2\pi\int_0^2 x(2x^2-x^3)\,dx$$

$$2\pi\int_0^2 (2x^3-x^4)\,dx$$

$$2\pi\left(\frac{1}{2}x^4-\frac{x^5}{5}\right)\Big|_0^2$$

$$2\pi\left(\frac{8}{1}-\frac{32}{5}\right)-0$$

$$\frac{16\pi}{5}u^3$$

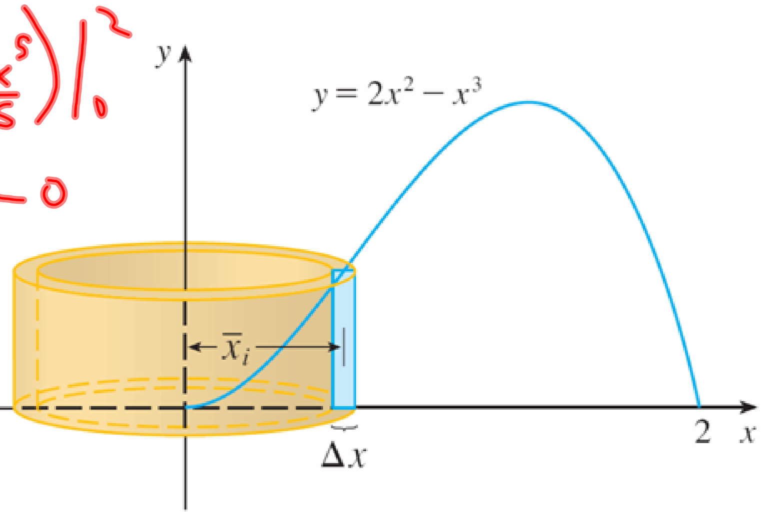


FIGURE 18 A cylindrical shell

$x=2y^2$ & $x=4+y^2$ Rotate about $y=-3$

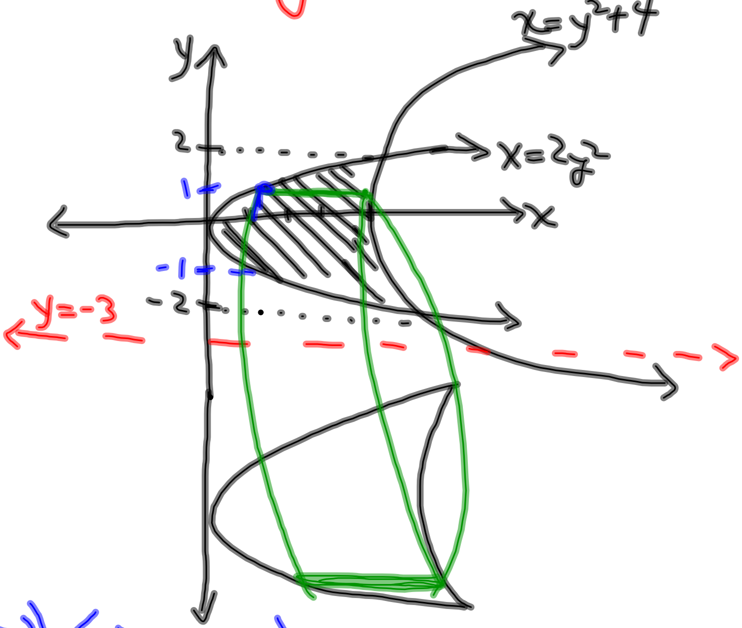
Volume?

$2y^2=4+y^2$

$y^2-4=0$

$(y-2)(y+2)=0$

$y=\pm 2$



$$V=2\pi\int_{-2}^{2}(y+3)(y^2+4-2y^2)\,dy$$

$$V=2\pi\int_{-2}^{2}(y+3)(4-y^2)\,dy$$

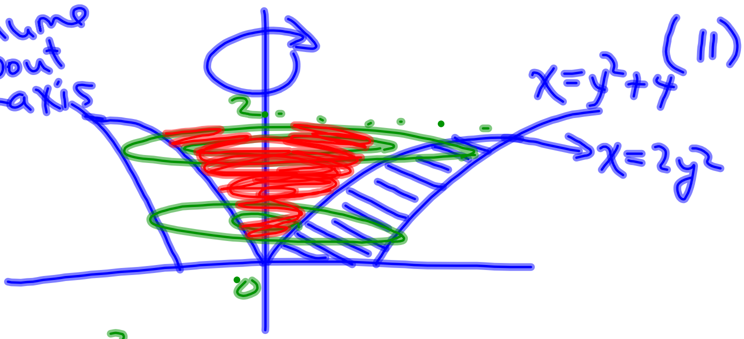
$$V=2\pi\int_{-2}^{2}(4y-y^3+12-3y^2)\,dy$$

$$=2\pi\left(2y^2-\frac{y^4}{4}+12y-y^3\right)\Big|_{-2}^{2}$$

$$=2\pi\left[(8-4+24-8)-(8-4-24+8)\right]$$

$$=\underline{64\pi\,u^3}$$

(i) Volume about y-axis



(ii) How much water would this object hold?

$$V=\pi\int_0^2 (y^2+4)^2-(2y^2)^2\,dy$$

$$V=\pi\int_0^2 (y^4+8y^2+16)-4y^4\,dy$$

$$V=\pi\int_0^2 (-3y^4+8y^2+16)\,dy$$

$$=\left(-\frac{3}{5}y^5+\frac{8}{3}y^3+16y\right)\pi\Big|_0^2$$

$$=\left(-\frac{96}{5}+\frac{64}{3}+\frac{32}{1}\right)\pi$$

$$=\left(\frac{-288+320+480}{15}\right)\pi$$

$$=\frac{512\pi}{15}\;u^3$$

(ii)

$$\pi\int_0^2 (2y^2)^2\,dy$$

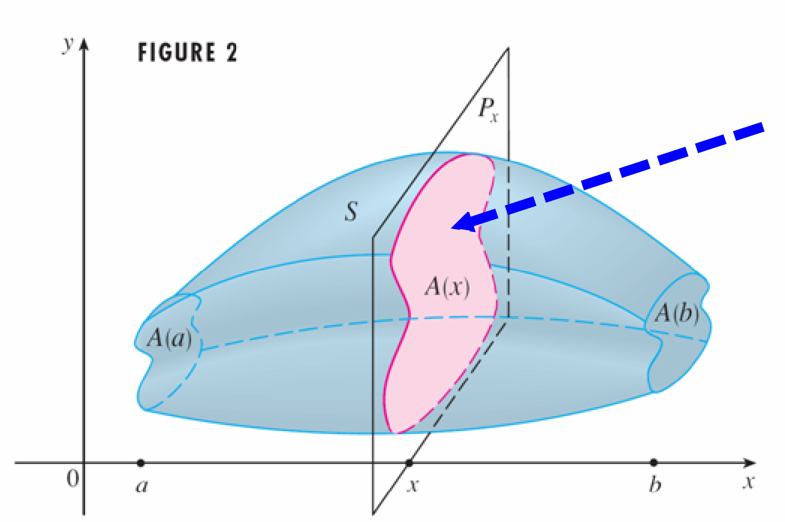
$$\pi\int_0^2 4y^4\,dy$$

$$\pi\left(\frac{4}{5}y^5\right)\Big|_0^2$$

$$=\frac{128\pi}{5}\;u^3$$

Volume: Using known cross-sectional areas

- The formula $V = \int_a^b A(x)\,dx$ can be applied to *any* solid for which the cross-sectional area $A(x)$ can be found

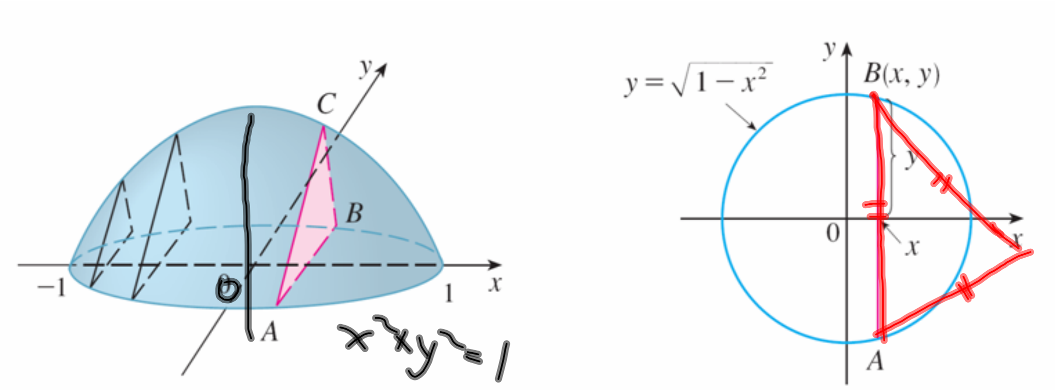


Notice that each cross-section would be the same shape. If we could determine the area of each of these cross-sections, we could determine the volume of this solid.

Think of it as summing the volume of each slice of bread to determine the volume of the loaf of bread!!

Example:

A solid has a circular base of radius 1 . Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

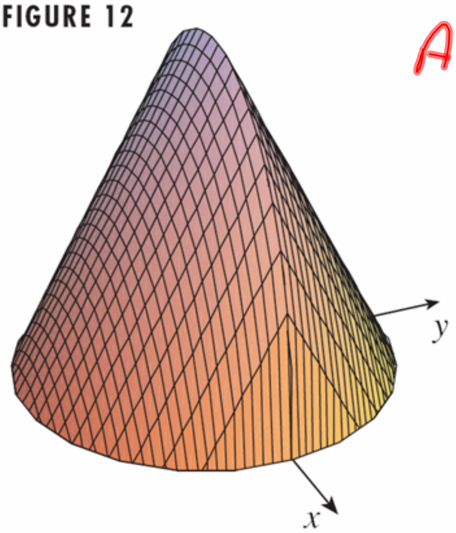


$x^2+y^2=1$

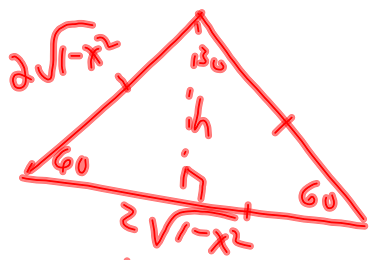
(a) The solid FIGURE 13 (b) Its base

$2\sqrt{1-x^2}$

FIGURE 12



$A = \frac{1}{2}bh$



$\sin 60° = \dfrac{h}{2\sqrt{1-x^2}}$

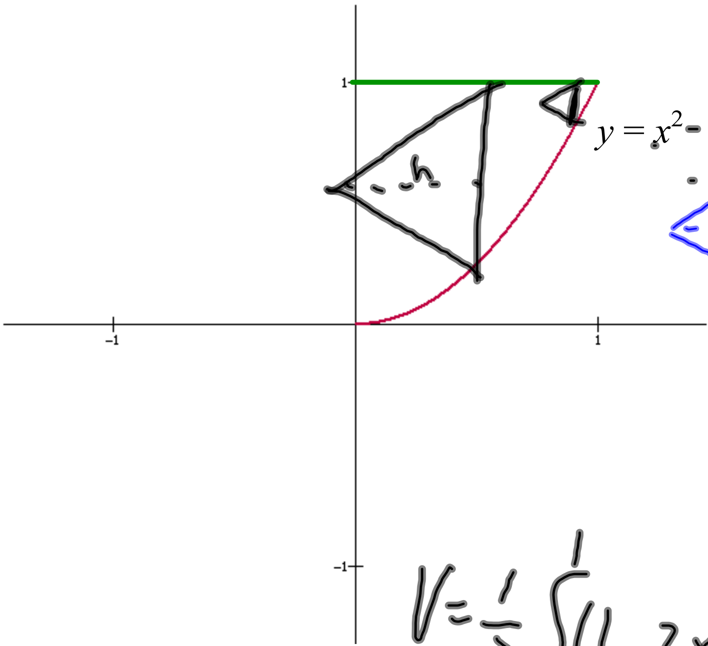
$\dfrac{\sqrt{3}}{2} = \dfrac{h}{2\sqrt{1-x^2}}$

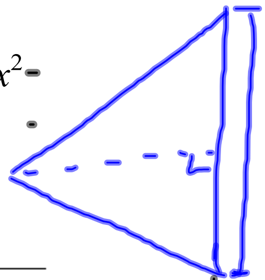
$$V = \frac{1}{2}\int_{-1}^{1} \underset{b}{\left(2\sqrt{1-x^2}\right)} \underset{h}{\left(\sqrt{3-3x^2}\right)}\,dx \qquad h = \sqrt{3}\cdot\sqrt{1-x^2} \quad \because(1-x^2)$$

$$V = \frac{1}{2}\int_{-1}^{1} 2\sqrt{1-x^2}\,(\sqrt{3})(\sqrt{1-x^2})\,dx$$

$$V = \sqrt{3}\int_{-1}^{1}(1-x^2)\,dx \quad \underline{\text{OR}} \quad 2\sqrt{3}\int_{0}^{1}(1-x^2)\,dx$$

Determine the volume of the solid S, whose base is the parabolic region $f(x) = x^2$ over the closed interval $0 \leq x \leq 1$, and whose cross-sections perpendicular to the x-axis are isosceles triangles with height equal to the base. solution: 4/15 u^3





base $= 1 - x^2$

$$V = \frac{1}{2}\int_0^1 (1-x^2)^2\,dx$$

$$V = \frac{1}{2}\int_0^1 (1-2x^2+x^4)\,dx$$

$$V = \frac{1}{2}\left(x - \frac{2}{3}x^3 + \frac{x^5}{5}\right)\Big|_0^1$$

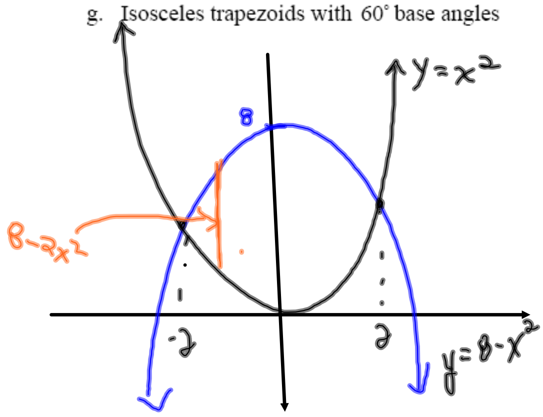
$$V = \frac{1}{2}\left[\left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0)\right]$$

$$= \frac{1}{2}\left(\frac{15-10+3}{15}\right)$$

$$= \frac{8}{30} = \frac{4}{15}$$

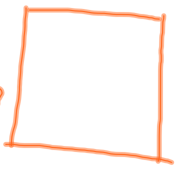
The base of the volume is the region bounded by the curves $y = 8 - x^2$ and $y = x^2$.
The cross sections perpendicular to the x-axis are:

a. Squares
b. Equilateral triangles
c. Isosceles right triangles with leg on the base
d. Isosceles right triangles with hypotenuse on the base
e. Semi-circles
f. Quarter-circles
g. Isosceles trapezoids with 60° base angles



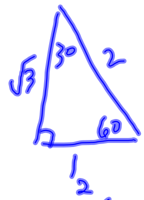
$x^2 = 8 - x^2$
$2x^2 = 8$
$x^2 = 4$
$x = \pm 2$

a)

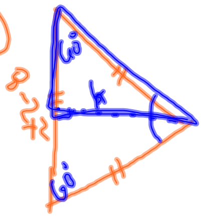


$V = \int_{-2}^{2} (8-2x^2)^2 \, dx$

$(8-x^2) - x^2$
$= 8-2x^2$



(b)



$\sin 60^\circ = \frac{h}{8-2x^2}$

$\frac{\sqrt{3}}{2} = \frac{h}{8-2x^2}$

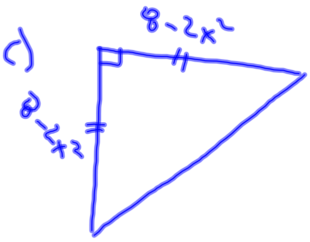
$h = \frac{\sqrt{3}}{2}(8-2x^2)$

$V = \frac{\sqrt{3}}{2} \int_{-2} (8-2x^2)(4-x^2) \, dx$

$h = \sqrt{3}(4-x^2)$

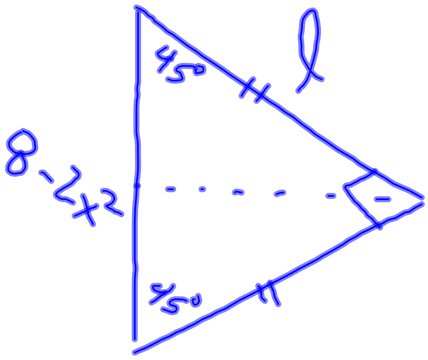
$(4-x^2)^2 + h^2 = (8-2x^2)^2$
$16 - 8x^2 + x^4 + h^2 = 64 - 32x^2 + 4x^4$
$h^2 = 3x^4 - 24x^2 + 48$
$h^2 = 3(x^4 - 8x^2 + 16)$
$h^2 = \sqrt{3}\sqrt{(x^2-4)^2}$
$h = \pm\sqrt{3}(x^2-4)$

c)

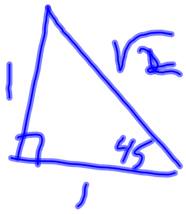


$V = \frac{1}{2} \int_{-2}^{2} (8-2x^2)^2 \, dx$

d)



$$\sin 45^\circ = \frac{l}{8-2x^2}$$



$$\frac{1}{\sqrt{2}} = \frac{l}{8-2x^2}$$

$$l = \frac{8-2x^2}{\sqrt{2}}$$

$$V = \frac{1}{2}\int_{-2}^{2}\left(\frac{8-2x^2}{\sqrt{2}}\right)^2 dx$$