Fundamental Theorem of Calculus

- The first part of the FTC deals with functions of the form

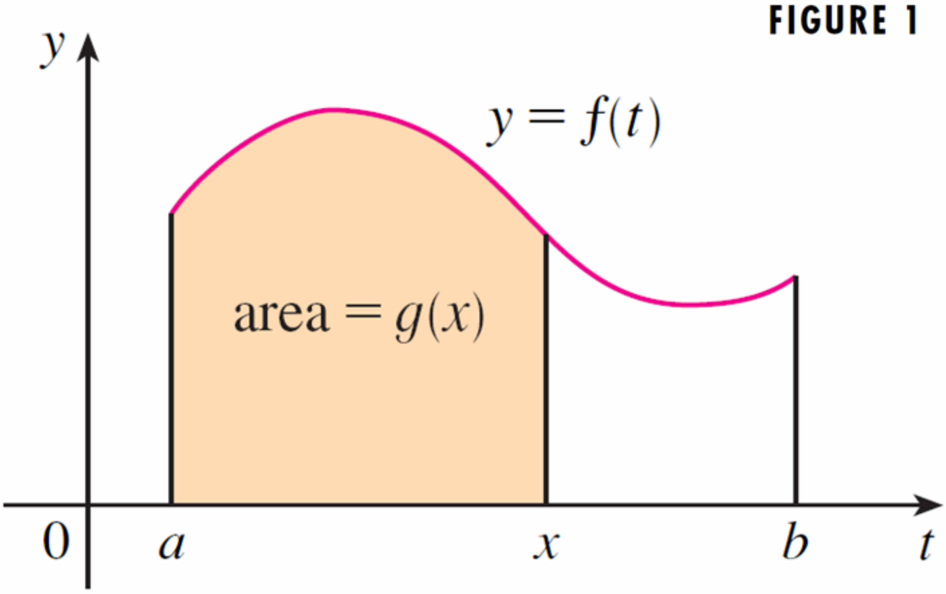
$$g(x) = \int_a^x f(t)\,dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b.

• If f happens to be a positive function, then what would $g(x)$ represent??

...the area under the graph of f from a to x, where x can vary from a to b.

FIGURE 1



If f is the function shown below and $g(x) = \int_a^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$.

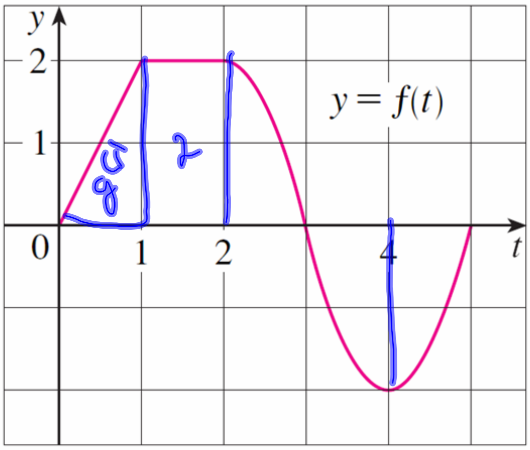


FIGURE 2

$g(0) = 0$
$g(1) = 1$
$g(2) = 3$
$g(3) \doteq 4.4$
$g(4) = 3$
$g(5) \approx 1.6$

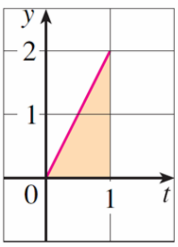
Let's look at g(0) and g(1)...

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 1 + (1 \cdot 2) = 3$$

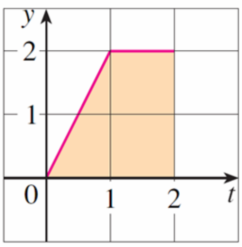
$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$

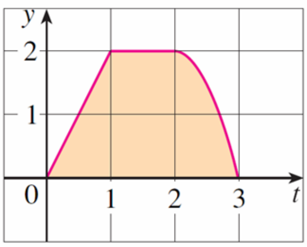
$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$



$g(1) = 1$



$g(2) = 3$



$g(3) \approx 4.3$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is an antiderivative of f, that is, $g'(x) = f(x)$ for $a < x < b$.

In Leibniz notation...

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

This is saying that integration and differentiation are inverses of one another.

Evaluate the following:

$$\frac{d}{dx}\int_1^x t^3\,dt$$

According to above, should equal?? x^3

Let's check using our traditional methods...

- Integrate and then differentiate

$$\left.\frac{t^4}{4}\right|_1^x$$

$$g(x) = \frac{x^4}{4} - \frac{1}{4}$$

$$g'(x) = x^3$$

More Examples:

$$\frac{d}{dx}\int_{-\pi}^{x}\cos t\,dt = \cos x$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

upper bound is not *x*...now what??

$$\frac{d}{dx}\int_1^{x^2} \cos t\,dt$$

Let $u = x^2$ and apply the chain rule when finding $\frac{dy}{dx}$ ...

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = \left(\frac{d}{du}\int_1^{u} \cos t\,dt\right) \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \bullet \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos x^2 \bullet 2x$$

Now let's try and [illegible] these a little quicker...

$$\frac{d}{dx}\int_0^{x^2} e^{t^2}\,dt = 2xe^{x^4}$$

$e^{(x^2)^2}(2x)$

$e^{x^4}(2x)$

$$\frac{d}{dx}\int_0^{5x} \frac{\sqrt{1+t^2}}{t}\,dt = \frac{5\sqrt{1+25x^2}}{5x} \quad (1/5)$$

Here are a couple with a little twist...

$$\frac{d}{dx}\int_x^5 3t\sin t\,dt = -3x\sin x$$

Lower bound is not a constant???

Compare these...

$$\int_1^3 x^2\,dx = \left.\frac{x^3}{3}\right|_1^3 = 9 - \frac{1}{3}$$

$$\int_3^1 x^2\,dx = \left.\frac{x^3}{3}\right|_3^1 = \frac{1}{3} - 9$$

$$\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt = \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$$

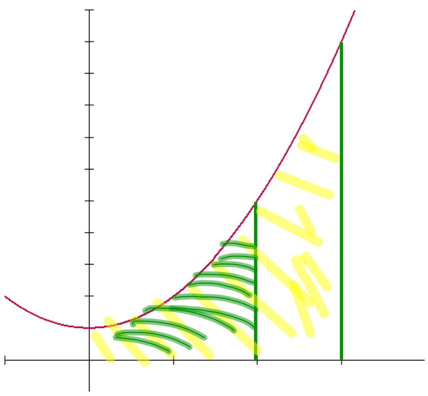
Neither bound is a constant???

$$\frac{d}{dx}\int_0^{x^2}\frac{1}{2+e^x}dt - \int_0^{2x}\frac{1}{2+e^x}dt$$

$$= \frac{1}{2+e^{x^2}}(2x) - \frac{1}{2+e^{2x}}(2)$$

Use this type of reasoning...

$$\int_1^3 (x^2+1)\,dx = \int_0^3 (x^2+1)\,dx - \int_0^1 (x^2+1)\,dx$$



Example:

Find $g(1)$, given that $g(x)=\frac{d}{dx}\int_{x^2}^{x^3}\left(3t-t^3\right)dt$

$$= \frac{d}{dx}\int_0^{x^3}\left(3t-t^3\right)dt - \frac{d}{dx}\int_0^{x^2}\left(3t-t^3\right)dt$$

$$g(x) = \left(3x^3-x^9\right)\left(3x^2\right) - \left(3x^2-x^6\right)\left(2x\right)$$

$$g(1) = (2)(3)-(2)(2)$$

$$= 6-4$$

$$= 2$$

Practice problems...

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#3, 7, 9, 11, 13, 15, 17, 21

Techniques of Integration

I. Substitution Technique

Examples:

$\int 3x(x^2+1)^{20}\,dx = \int 3\left(\frac{du}{2}\right)u^{20}$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

$= \frac{3}{2}\int u^{20}\,du$

$= \frac{3}{2(21)}u^{21} + C$

$= \frac{1}{14}u^{21} + C$

$= \frac{1}{14}(x^2+1)^{21} + C$

~~$u = (x^2+1)^{20}$~~

~~$\frac{du}{dx} = 20(x^2+1)^{19}(2x)$~~

$\int \sin^2 x \cos x\,dx$

$u = \sin x$

$du = \cos x\,dx$

$\int u^2\,du$

$= \frac{u^3}{3} + C$

$= \frac{\sin^3 x}{3} + C$

$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2}x^{-1/2}\,dx$

$2\,du = \frac{dx}{\sqrt{x}}$

$\int e^u(2\,du)$

$2\int e^u\,du$

$= 2e^u + C$

$= 2e^{\sqrt{x}} + C$

$2\int e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}\,dx$ ($e^u \cdot du$)

$= 2e^{\sqrt{x}} + C$

$\int x^2 \sqrt{x-1}\,dx$ ← ???
This one is a little different

$\int_0^2 x^2 \sqrt{x^3+1}\,dx$ ← What about a definite integral???

Look at both methods....

$$\int_{e^{-6}}^{e^{6}} \frac{5+\ln^3 x}{x}\,dx$$

Attachments

Worksheet - Nature of the Roots.doc