TEST: EXTREME VALUES

NAME: Solutions

MAY 2012

1. Given the function $f(x) = 2x^3 - 3x^2 - 36x + 14$ determine ... [10]
   (a) the absolute maximum and minimum values on the interval [0 , 4].
   (b) the intervals of increase/decrease and local extreme values on the interval (-∞ , ∞).

(a) $f'(x) = 6x^2 - 6x - 36$

$0 = x^2 - x - 6$

$0 = (x-3)(x+2)$

$x = 3, -2$

x	y
0	14
3	-67
4	-50

$\boxed{\begin{array}{l}\text{Abs. Max} = 14 \\ \text{Abs. Min.} = -67\end{array}}$

b)

	$x-3$	$x+2$	f'	f
$(-\infty, -2)$	−	−	+	Inc
$(-2, 3)$	−	+	−	Dec
$(3, \infty)$	+	+	+	Inc

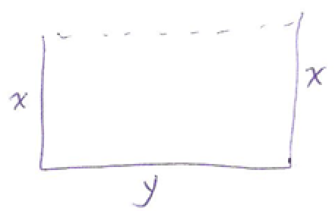
Increase: $(-\infty, -2)$, $(3, \infty)$

Decrease: $(-2, 3)$

Local Max.: $(-2, 58)$

Local Min.: $(3, -67)$

2. A field of rectangular shape is to be fenced off along the bank of a river. No fence is required on the side lying along the river. If the material for the fence costs \$2/m for the two ends, and \$3/m for the side parallel to the river, find the dimensions of the field of maximum area that can be enclosed with \$900 worth of fencing. [6]



$(2)2x + y(3) = 900$

$4x + 3y = 900$

$3y = 900 - 4x$ $= \frac{900-4x}{3}$

$y = 300 - \frac{4}{3}x$

$A = xy$

$A = x\left(300 - \frac{4}{3}x\right)$

$A = 300x - \frac{4}{3}x^2$

$A' = 300 - \frac{8}{3}x$

$0 = 300 - \frac{8}{3}x$

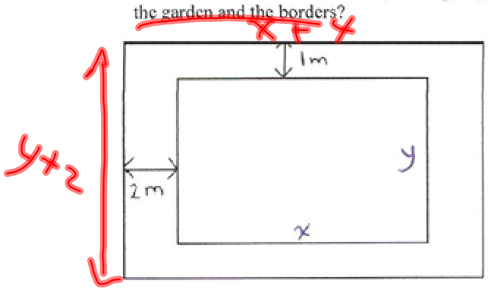
$\frac{8}{3}x = 300$

$x = \frac{900}{8}$

$\underline{x = 112.5\text{ m}}$ & $y = 300 - \frac{4}{3}(112.5)$

$y = \underline{150\text{ m}}$

3. A rectangular flower garden with an area of 30 m^2 is surrounded by a grass border 1 m wide on two sides and 2 m wide on the other two sides (see diagram). What dimensions of the garden minimize the combined area of the garden and the borders? [6]



$xy = 30$

$y = \frac{30}{x}$

$A = (x+4)(y+2)$

$A = (x+4)\left(\frac{30}{x}+2\right)$

$A = 30 + 2\cancel{x} + \frac{120}{x} + 8$

$A = 38 + 2x + 120x^{-1}$

$A' = 2 - 120x^{-2}$

$0 = 2 - \frac{120}{x^2}$

$0 = 2x^2 - 120$

$120 = 2x^2$

$60 = x^2$

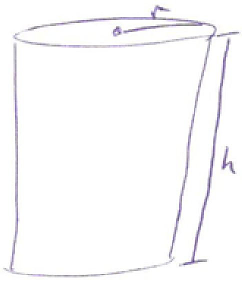
$x = \sqrt{60}$

$x = 7.75m$

$y = \frac{30}{\sqrt{60}} = 3.87m$

4. A closed cylindrical can is to be constructed with volume 1 m^3. Material for the top and bottom costs 80 cents/m^2, while material for the curved sides costs 50 cents/m^2. Determine the dimensions of the can (radius and height) that will minimize the total cost of constructing this can. [6]

(You may find the following formulas useful. The area of a circle with radius r is πr^2. The volume of a cylinder with radius r and height h is $\pi r^2 h$. The surface area of a cylinder, excluding the top and bottom is $2\pi r h$.)



$\pi r^2 h = 1$

$h = \frac{1}{\pi r^2}$

$Cost = 80(2\pi r^2) + 50(2\pi r h)$

$C = 160\pi r^2 + 100\pi r\left(\frac{1}{\pi r^2}\right)$

$C = 160\pi r^2 + 100 r^{-1}$

$C' = 320\pi r - 100 r^{-2}$

$320\pi r = \frac{100}{r^2}$

$r^3 = \sqrt[3]{\frac{100}{320\pi}}$

$r = 0.463m$

$height = \frac{1}{\pi(0.463)^2}$

$h = 1.483m$

5. While in refrigerated storage 1000m^3 of apples spoil at the rate of 20m^3/month. The September price is \$2.50/m^3, and the price increases \$1.25/m^3 each month. Storage costs are \$750/month. If the apples are placed in storage in September, when should they be sold to maximize profit ? [6]

$x \to$ Rep. Months after Sept.

$$\text{Profit} = (1000 - 20x)(2.50 + 1.25x) - 750x$$

$$P = 2500 + 1250x - 50x - 25x^2 - 750x$$

$$P = -25x^2 + 450x + 2500$$

$$P' = -50x + 450$$

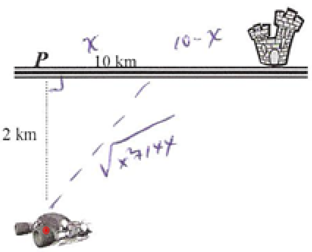
$$-450 = -50x$$

$$\underline{9 = x}$$

9 Months later (June)

6. A Sheik is traveling in a dune buggy in the desert 12 km due south of the nearest point **P** on a straight east-west road. The Sheik is trying to get to an Emperor's castle that is located along the road at a point 10 km east of point P. His dune buggy averages 15 km/h over the sand, and 39 km/h traveling along the road. Where should the Sheik head for along the road in order to get to the Emperor's castle as quickly as possible? [6]

$t = \frac{d}{s}$



$$\text{Time} = \frac{\sqrt{x^2+144}}{15} + \frac{10-x}{39}$$

$$T = \frac{1}{15}(x^2+144)^{1/2} + \frac{10}{39} - \frac{1}{39}x$$

$$T' = \frac{1}{30}(x^2+144)^{-1/2}(2x) - \frac{1}{39}$$

$$0 = \frac{2x}{30\sqrt{x^2+144}} - \frac{1}{39}$$

$$\frac{1}{39} = \frac{2x}{30\sqrt{x^2+144}}$$

$$30\sqrt{x^2+144} = 78x$$

$$900(x^2+144) = 6084x^2$$

$$129600 = 5184x^2$$

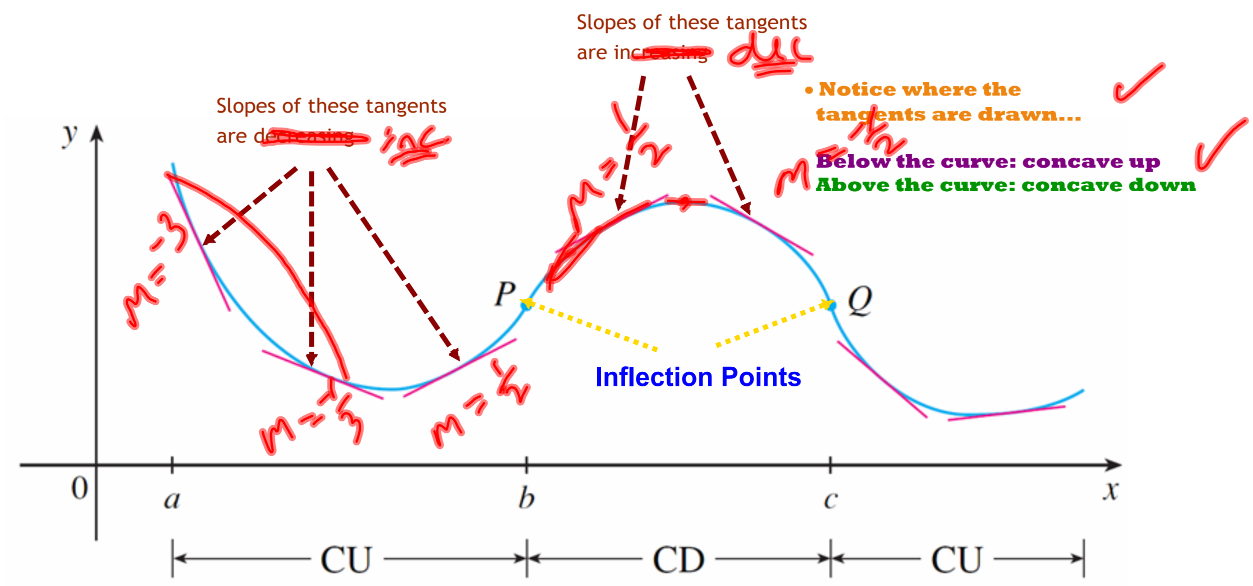
$$25 = x^2$$

$$5 = x$$

Should head for a point 5km east of P.

Concavity

A function (or its graph) is called **concave upward** on an interval I if f' is an increasing function on I. It is called **concave downward** on I if f' is decreasing on I.



- A point where a curve changes its direction of concavity is called an ***inflection point***.

If $f'(x) > 0$ then $f(x)$ is increasing,
so if $f''(x) > 0$ then $f'(x)$ is increasing.

F'' Pos → UP
F'' Neg → DN

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Thus there is a point of inflection at any point where the second derivative changes sign.

Second Derivative Test for Local Extrema

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

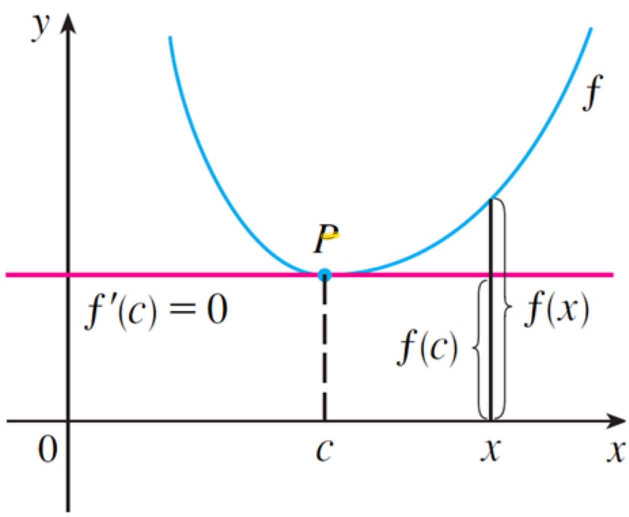


FIGURE 6
$f''(c) > 0$, f is concave upward

Example:

Examine the function $f(x) = x^4 - 4x^3$ with respect to…

- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

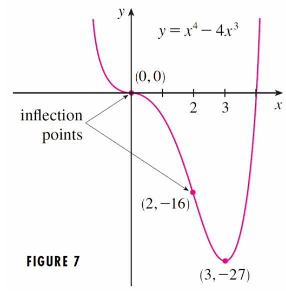


FIGURE 7

Inc/Dec.

f' crit. values: $4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0, 3$

	$4x^2$	$x-3$	f'	f
$(-\infty, 0)$	+	−	−	Dec
$(0, 3)$	+	−	−	Dec
$(3, \infty)$	+	+	+	Inc

Local Max.

None

Local Min.

$(3, -27)$

Concavity:

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0, 2$

	$12x$	$x-2$	f''	f concavity
$(-\infty, 0)$	−	−	+	Up
$(0, 2)$	+	−	−	Down
$(2, \infty)$	+	+	+	Up

Inflection Points:

$(0,0)$ $(2,-16)$

x-Intercepts

$x^4 - 4x^3 = 0$

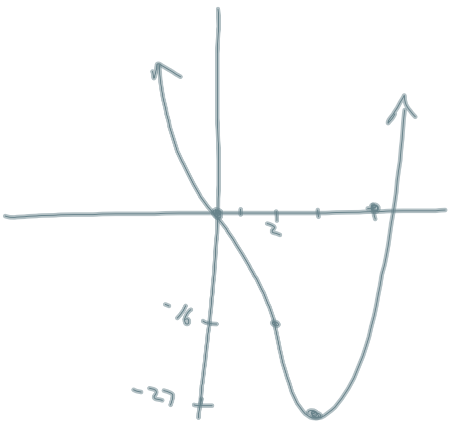
$x^3(x-4) = 0$

$x = 0, 4$

y-Intercepts $(x=0)$

$y = 0 - 0$

$y = 0$



Example:

Using the function: $f(x) = \dfrac{x^2}{x-7}$

Determine each of the following...

- Intercepts
- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

$$f'(x) = \frac{2x(x-7) - x^2(1)}{(x-7)^2}$$

$$f'(x) = \frac{2x^2 - 14x - x^2}{(x-7)^2}$$

$$f'(x) = \frac{x^2 - 14x}{(x-7)^2}$$

$$\boxed{f'(x) = \frac{x(x-14)}{(x-7)^2}}$$

$$f''(x) = \frac{(2x-14)(x-7)^2 - (x^2-14x)\left[2(x-7)\right]}{(x-7)^4}$$

$$f''(x) = \frac{\cancel{(x-7)}\left[2(x-7)(x-7) - (x^2-14x)(2)\right]}{(x-7)^{3}}$$

$$f''(x) = \frac{2(x-7)^2 - 2x^2 + 28x}{(x-7)^3}$$

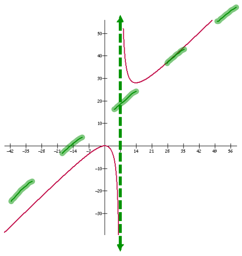
$$f''(x) = \frac{2(x^2-14x+49) - 2x^2 + 28x}{(x-7)^3}$$

$$\boxed{f''(x) = \frac{98}{(x-7)^3}}$$

Let's look at homework question...

Example:

Using the function: $f(x) = \dfrac{x^2}{x-7} = \dfrac{14^2}{7}$



Determine each of the following...

- Intercepts
- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

$f'(x) = \dfrac{x(x-14)}{(x-7)^2}$ $\qquad f''(x) = \dfrac{98}{(x-7)^3}$

Intercepts:

x-Int. $(y=0)$

$0 = \dfrac{x^2}{(x-7)}$

$x^2 = 0$

$x = 0$

y-Int. $(x=0)$

$y = \dfrac{0^2}{0-7}$

$y = 0$

$(0,0)$

Max/Min.

$f'(x) = \dfrac{x(x-14)}{(x-7)^2}$

Critical Values

$x = 0, 14, 7$

	x	$x-14$	$(x-7)^2$	f'	f
$(-\infty,0)$	$-$	$-$	$+$	$+$	Inc
$(0,7)$	$+$	$-$	$+$	$-$	Dec
$(7,14)$	$+$	$-$	$+$	$-$	Dec
$(14,\infty)$	$+$	$+$	$+$	$+$	Inc

Local Max.

$(0,0)$

Local Min.

$(14,28)$

Concavity

$f''(x) = \dfrac{98}{(x-7)^3}$

C. Value $\Rightarrow x = 7$

	98	$(x-7)^3$	f''	f
$(-\infty,7)$	$+$	$-$	$-$	Down
$(7,\infty)$	$+$	$+$	$+$	Up

Inflection Point: None

$(7, \ )$ undefined

Asymptotes:

Horizontal:

$F(x) = \dfrac{x^2}{x-7}$

$\lim_{x\to\infty} \dfrac{\frac{x^2}{x^2}}{\frac{x}{x^2} - \frac{7}{x^2}}$

$= \dfrac{1}{0-0}$

None

Vertical: (Set Den. $=0$)

$x - 7 = 0$

$x = 7$

$\lim_{x\to 7^-} \dfrac{x^2}{x-7} = \dfrac{49}{\text{small}(-)} \to -\infty$

$\lim_{x\to 7^+} \dfrac{x^2}{x-7} = \dfrac{49}{\text{small}(+)} \to \infty$

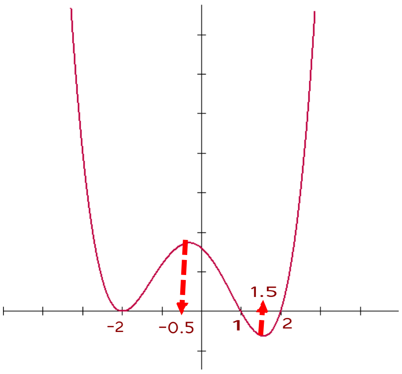
Example:

Sketch the function $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$. Use the following to help with the sketch:

- Intercepts
- Intervals of increase/decrease
- Concavity
- Points of inflection
- Local maximum and minimum values

Warm-Up

The graph of the derivative of a function f on the interval [-4 , 4] is shown below:



(a) On what intervals is f increasing?

(b) On what intervals is the graph of f concave up?

(c) At what x-coordinate does f have local extrema?

(d) What are the x-coordinates of all inflection points of the graph of f?

Determine the regions of concavity and all inflection points for the following function: $f(x) = x^4 - 2x^3 - 12x^2 + 3$

Asymptotes

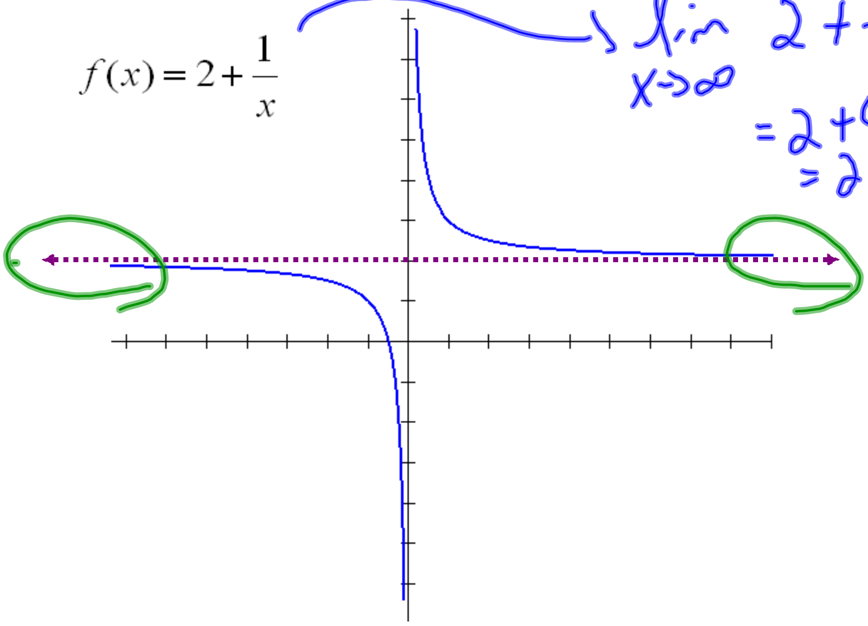
Horizontal Asymptote

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b$$

$$f(x) = 2 + \frac{1}{x}$$

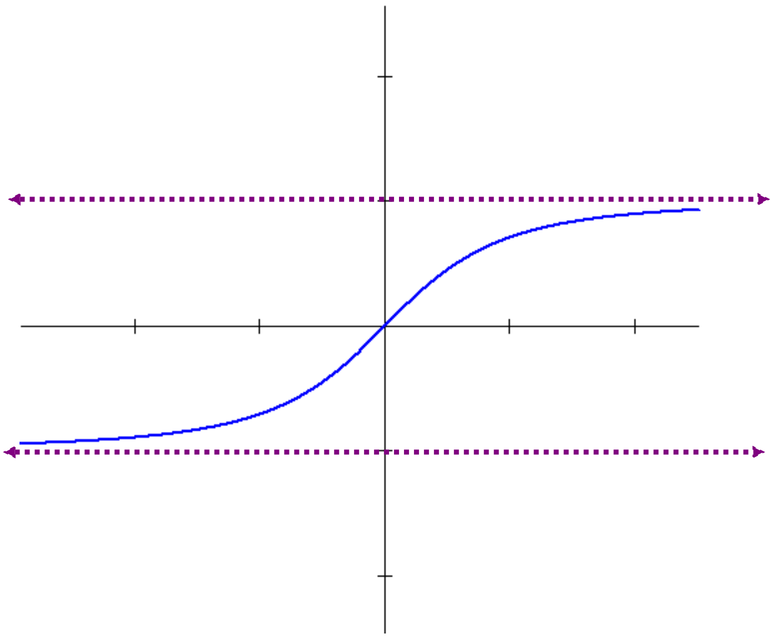
$$\lim_{x\to\infty} 2 + \frac{1}{x} = 2 + 0 = 2 \qquad \lim_{x\to-\infty} 2 + \frac{1}{x} = 2$$



Examine the limits of $f(x)$ as x approaches $\pm\infty$

There can be more than one horizontal asymptote.

Examine the function $f(x) = \dfrac{x}{\sqrt{x^2+1}}$



Examine the limits of $f(x)$ as x approaches $\pm\infty$

Vertical Asymptote

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^-} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^+} f(x) = \pm\infty$$

* Any value that make f(x) undefined

Example: $f(x) = \dfrac{1}{x^2 - x - 2}$

Vertical (Den = 0)

$x^2 - x - 2 = 0$

$(x-2)(x+1) =$

$x = 2, -1$

$\lim_{x\to -1^-} \dfrac{1}{(x-2)(x+1)}$

$= \dfrac{1}{(-1-2)(-1.000..1+1)}$

$= \dfrac{1}{\text{small}(+)}$

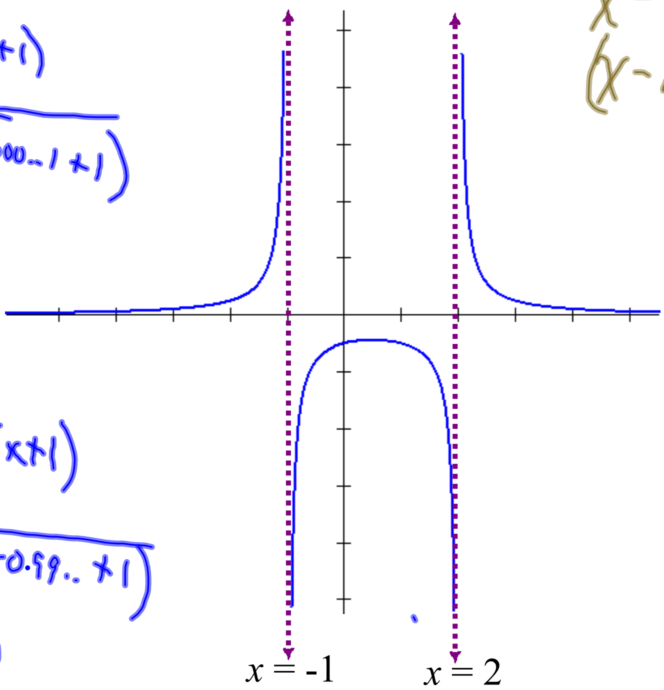
$\to \infty$

$\lim_{x\to -1^+} \dfrac{1}{(x-2)(x+1)}$

$= \dfrac{1}{(-1-2)(-0.99..+1)}$

$= \dfrac{1}{\text{small}(-)}$

$\to -\infty$



Use limits to examine the behaviour of the function near the asymptotes

Sketch the following function:

$$f(x)=\frac{8(x-2)}{x^2} \qquad f'(x)=\frac{-8(x-4)}{x^3} \qquad f''(x)=\frac{16(x-6)}{x^4}$$

Be sure to examine...

- Intercepts
- Asymptotes (*vertical and horizontal*)
- Regions of increase/decrease
- Local extrema
- Regions where concave up/down
- Inflection points

Intercepts

x-Int.

$0=\frac{8(x-2)}{x^2}$

$x=2$

$(2,0)$

y-Int.

$y=\frac{8(0-2)}{0^2}$

undefined

$\therefore$ None

Asymptotes

Horizontal

$\lim_{x\to\pm\infty}\frac{\frac{8x}{x^2}-\frac{16}{x^2}}{\frac{x^2}{x^2}}$

$=\frac{0-0}{1}$

$\boxed{y=0}$

Vertical

$x^2=0$

$x=0$

$\lim_{x\to 0^-}\frac{8(x-2)}{x^2}$

$\to -\infty$

$\lim_{x\to 0^+}\frac{8(x-2)}{x^2}$

$\to -\infty$

Inc/Dec.

$f'(x)=\frac{-8(x-4)}{x^3}$

Critical Values:

$x=4,0$

	-8	$x-4$	x^3	f'	f
$(-\infty,0)$	$-$	$-$	$-$	$-$	Dec
$(0,4)$	$-$	$-$	$+$	$+$	Inc
$(4,\infty)$	$-$	$+$	$+$	$-$	Dec

Local Max.

$(4,1)$

Local Min.

None

Concavity

$0=\frac{16(x-6)}{x^4}$

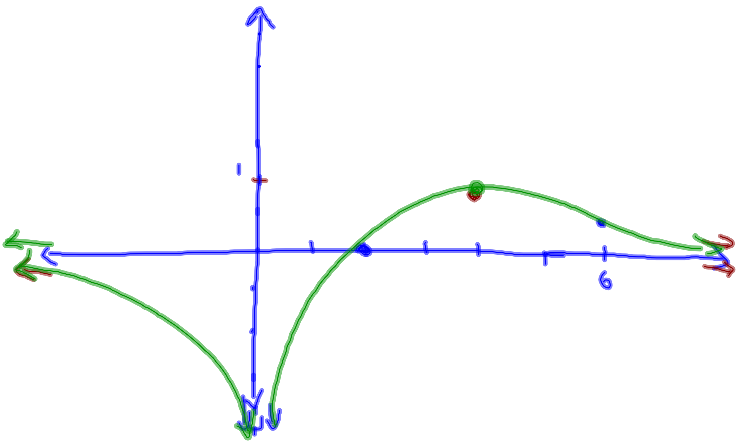
Critical Values

$x=6,0$

	16	$x-6$	x^4	f''	f
$(-\infty,0)$	$+$	$-$	$+$	$-$	Down
$(0,6)$	$+$	$-$	$+$	$-$	Down
$(6,\infty)$	$+$	$+$	$+$	$+$	Up

Inflection Point(s)

$\left(6,\frac{8}{9}\right)$



Given $f(x) = x^{1/3}(4 + x)$,

Intercepts

$$f'(x) = \frac{4x + 4}{3x^{2/3}} \qquad \text{and} \qquad f''(x) = \frac{4x - 8}{9x^{5/3}}.$$

(a) Find and specify all intervals where f is increasing; decreasing; concave up; and concave down.

(b) Determine the coordinates of any relative extreme values and any points of inflection.

(c) Sketch a graph of f, showing all information obtained in parts (a) and (b).

Calculus 120
Test : Curve Sketching

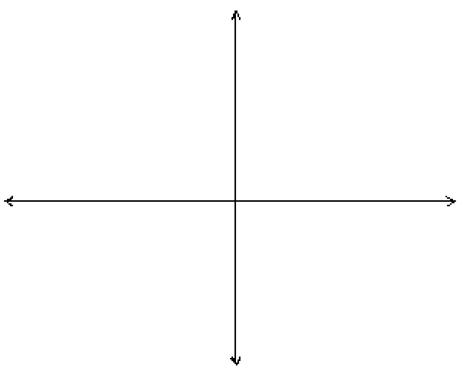
1. Consider the function : $f(x)=\dfrac{4\left(x^2-x-2\right)}{(x+2)^2}$ ***(value = 20)***

given $f'(x)=\dfrac{4(5x+2)}{(x+2)^3}$ and $f''(x)=\dfrac{-8(5x-2)}{(x+2)^4}$

Supply the information requested in the boxes at right and give a careful sketch of f on the axes below.

(***Note:*** $f\left(-\frac{2}{5}\right)\approx -2.25$ *and* $f\left(\frac{2}{5}\right)\approx -1.56$)

x-intercept(s)
y-intercept(s)
Vertical asymptote(s)
Horizontal asymptote(s)
Region(s) of increase
Region(s) of decrease
Local maxima
Local minima
Region(s) where concave up
Region(s) where concave down
Point(s) of inflection



Calculus 120
Test : Curve Sketching

1. Consider the function: $f(x) = \frac{8(x-2)}{x^2}$ **(value = 20)**

given $f'(x) = \frac{-8(x-4)}{x^3}$ and $f''(x) = \frac{16(x-6)}{x^4}$

Supply the information requested in the boxes at right and give a careful sketch of f on the axes below.

x-intercept(s): $(2,0)$
y-intercept(s): None
Vertical asymptote(s): $x=0$
Horizontal asymptote(s): $y=0$
Region(s) of increase: $(0,4)$
Region(s) of decrease: $(-\infty,0)$ & $(4,\infty)$
Local maxima: $(4,1)$
Local minima: None
Region(s) where concave up: $(6,\infty)$
Region(s) where concave down: $(-\infty,6)$ $(-\infty,0)$ $(0,6)$
Point(s) of inflection: $(6,\frac{8}{9})$

x-Int. $(y=0)$

$0=\frac{8(x-2)}{x^2}$

$x=2$

y-Int. $(x=0)$

$\frac{8(0-2)}{0^2}$ = undefined

$\therefore$ None

Asymptotes.

Vertical (set den. = 0)

$x^2=0$

$x=0$

Horizontal $(x \to \pm\infty)$

$\lim_{x\to\pm\infty} \frac{\frac{8x}{x^2}-\frac{16}{x^2}}{\frac{x^2}{x^2}} = \frac{0}{1}$

$y=0$

Inc/Dec

$\frac{-8(x-4)}{x^3}=0$

Critical Values:

$x=4,0$

	-8	$x-4$	x^3	f'	f
$(-\infty,0)$	$-$	$-$	$-$	$-$	Dec
$(0,4)$	$-$	$-$	$+$	$+$	Inc
$(4,\infty)$	$-$	$+$	$+$	$-$	Dec

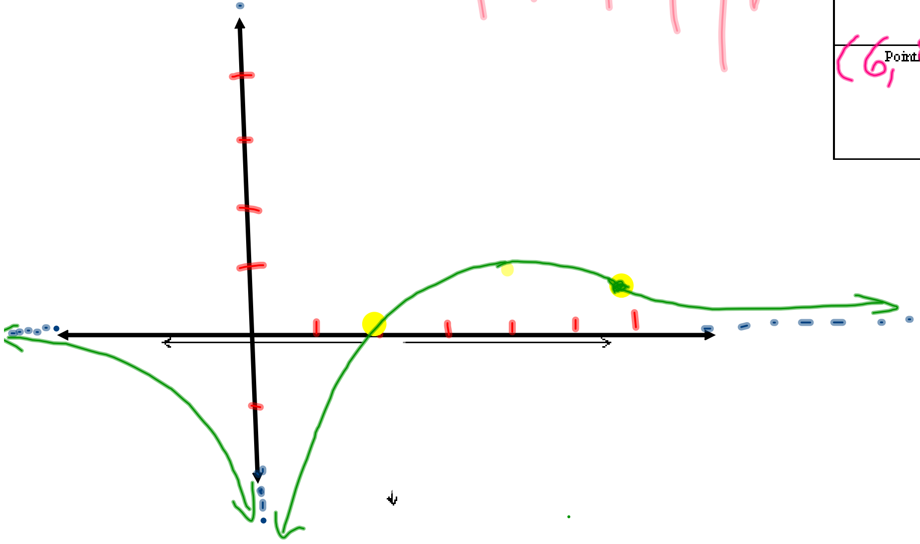
Concavity

$\frac{16(x-6)}{x^4}=0$

Critical Values:

$x=6,0$

	16	$x-6$	x^4	f''	f
$(-\infty,0)$	$+$	$-$	$+$	$-$	Down
$(0,6)$	$+$	$-$	$+$	$-$	Down
$(6,\infty)$	$+$	$+$	$+$	$+$	Up



$f(x) = x^4 - 2x^2$ $f'(x) = 4x^3 - 4x = 4x(x^2-1)$ $f''(x) = 12x^2 - 4$

2. Consider the function: $f(x) = x^4 - 2x^2$ ***(value = 20)***

Supply the information requested in the boxes at right and provide a careful sketch of f on the axes below.

Note: $f\left(\pm\frac{1}{\sqrt{3}}\right) \approx -\frac{5}{9}$

x-intercept(s)	$(0,0)$ $(\sqrt{2},0)$ $(-\sqrt{2},0)$
y-intercept(s)	$(0,0)$
Region(s) of increase	$(-1,0)$ & $(1,\infty)$
Region(s) of decrease	$(-\infty,-1)$ & $(0,1)$
Local maxima	$(0,0)$
Local minima	$(-1,-1)$ $(1,-1)$
Region(s) where concave up	$(-\infty,-\frac{1}{\sqrt{3}})$ & $(\frac{1}{\sqrt{3}},\infty)$
Region(s) where concave down	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$
Point(s) of inflection	$(-\frac{1}{\sqrt{3}}, -0.6)$ $(\frac{1}{\sqrt{3}}, -0.6)$

≈ -0.5

Intercepts

x-Int $(y=0)$

$0 = x^4 - 2x^2$

$0 = x^2(x^2-2)$

$0 = x^2(x-\sqrt{2})(x+\sqrt{2})$

$x = 0, \pm\sqrt{2}$

y-Int. $(x=0)$

$y = 0^2 - 0$

$y = 0$

Inc/Dec

$4x(x^2-1) = 0$

$4x(x-1)(x+1) = 0$

$x = 0, \pm 1$

	$4x$	$x-1$	$x+1$	f'	f
$(-\infty,-1)$	−	−	−	−	Dec
$(-1,0)$	−	−	+	+	Inc
$(0,1)$	+	−	+	−	Dec
$(1,\infty)$	+	+	+	+	Inc

Concavity

$12x^2 - 4 = 0$

$4(3x^2-1) = 0$

$4(\sqrt{3}x-1)(\sqrt{3}x+1)$

$x = \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

	4	$\sqrt{3}x-1$	$\sqrt{3}x+1$	f''	f
$(-\infty,-\frac{1}{\sqrt{3}})$	+	−	−	+	Up
$(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$	+	−	+	−	Down
$(\frac{1}{\sqrt{3}},\infty)$	+	+	+	+	Up