How do we determine absolute maximum and minmum values?

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example:

Given the function...

$$f(x) = 3x^4 - 16x^3 + 18x^2 \ , \ -1 \le x \le 4$$

Determine the absolute maximum and minimum values of the function.

Solution

$$f'(x) = 12x^3 - 48x^2 + 36x$$

Critical Values:

$$0 = 12x(x^2 - 4x + 3)$$
$$0 = 12x(x-3)(x-1)$$

$$x = 0, 3, 1 \qquad [-1, 4]$$

	x	y
start →	-1	37
C.V. values	0	0
	1	5
	3	-27
end →	4	32

Absolute Max. = 37

Absolute Min. = -27

Example 2:

Using Calculus methods determine the absolute maximum and minimum values of the function given below:

$f(x) = x^3 + 2x^2 + x - 1$ over the interval [-1 , 1]

$f'(x) = 3x^2 + 4x + 1$

$0 = 3x^2 + 3x + x + 1$

$0 = 3x(x+1) + 1(x+1)$

$0 = (x+1)(3x+1)$

$x = -1, -\frac{1}{3}$

x	y
-1	-1
$-\frac{1}{3}$	-1.148
1	3

Abs. Min.

Abs. Max.

Warm-Up

Given the function $f(x) = 4x^4 - 8x^2 + 1$ determine ...
(a) the absolute maximum and minimum values on the interval [0 , 3].
(b) the intervals of increase/decrease and local extreme values on the interval $(-\infty, \infty)$.

a) $f'(x) = 16x^3 - 16x$

$0 = 16x(x^2 - 1)$

$0 = 16x(x-1)(x+1)$

$x = 0, \pm 1$

x	y
0	1
1	-3 ← Abs. Min.
3	253 ← Abs. Max.

b)

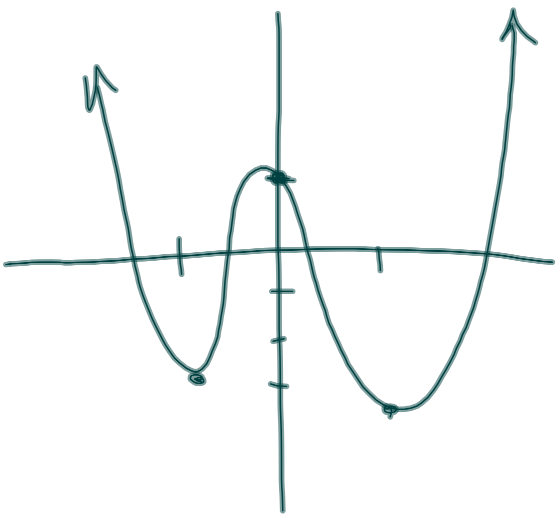
	16x	x−1	x+1	f′	f
$(-\infty, -1)$	−	−	−	−	Dec
$(-1, 0)$	−	−	+	+	Inc
$(0, 1)$	+	−	+	−	Dec
$(1, \infty)$	+	+	+	+	Inc

LOCAL MAX

(0, 1)

LOCAL MIN

(−1, −3) & (1, −3)



Optimization Problems

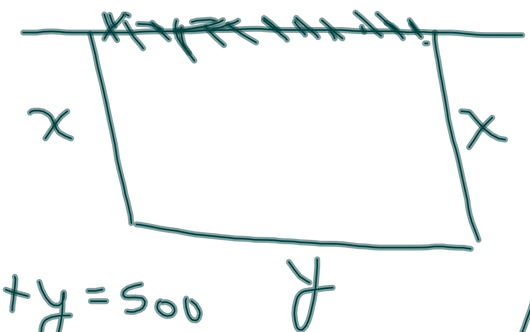
In optimization problems, in general, we are looking for the largest and/or smallest that a function can be. We saw how to one kind of optimization problem in the Absolute Extrema section where we found the largest and smallest value that a function would take on an interval.

In this section we are going to look at another type of optimization problem. Here we will be looking for the largest or smallest values of a function subject to some kind of constraint. It's usually easiest to see how these work with some examples.

Example 1:

We need to enclose a field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

- Draw a sketch
- Determine the constraint
- Determine a function in terms of a single variable
- Determine the absolute maximum value of this function
- Test solution with second derivative



$2x + y = 500$

$y = 500 - 2x$

$A = xy$

$A = x(500 - 2x)$

$A = 500x - 2x^2$

$A' = 500 - 4x$

$0 = 500 - 4x$

$4x = 500$

$x = 125$

$y = 500 - 2(125)$

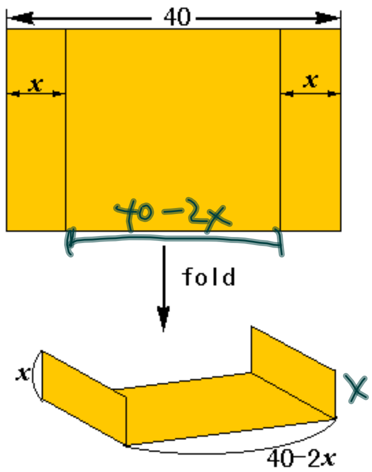
$y = 250$

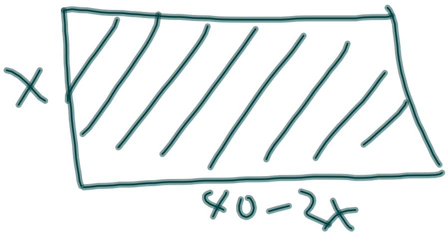
125 ft. x 250 ft.

Example 2:

A piece of tin 40 cm wide is to be folded up as shown.

How deep will this gutter be if it is to have maximum carrying capacity?





$A = x(40-2x)$

$A = 40x - 2x^2$

$A' = 40 - 4x$

$0 = 40 - 4x$

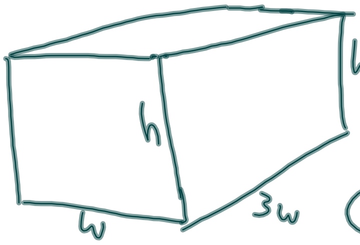
$4x = 40$

$x = 10$

$\therefore$ 10 cm deep

Example 3:

We are going to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50ft^3 determine the dimensions that will minimize the cost to build the box.



Top & Bottom

$$C = 10(6w^2) + 6(6wh) + 6(2wh)$$

$$C = 60w^2 + 36wh + 12wh$$

$$C = 60w^2 + 48w^1\left(\frac{50}{3w^2}\right)$$

$$C = 60w^2 + 800w^{-1}$$

$$C' = 120w - 800w^{-2}$$

$$V = 3w^2h$$

$$50 = 3w^2h$$

$$h = \frac{50}{3w^2}$$

Minimum Value $\longrightarrow$ $0 = 120w - \frac{800}{w^2}$

$$0 = 120w^3 - 800$$

$$120w^3 = 800$$

$$w^3 = \frac{800}{120} = \frac{20}{3}$$

$$w = \sqrt[3]{\frac{20}{3}} = 1.88 \text{ ft.}$$

$$l = 3w$$
$$= 3(1.88)$$
$$= 5.64 \text{ feet}$$

$$h = \frac{50}{3w^2} = \frac{50}{3(1.88)^2} = 4.72 \text{ feet}$$

(b)

What is the minimum cost?

$$C = 60w^2 + \frac{800}{w}$$

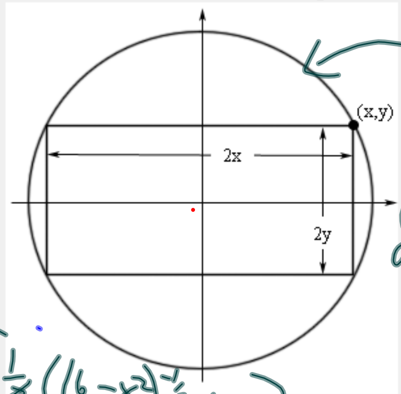
$$C = 60(1.88)^2 + \frac{800}{1.88}$$

$$C = \$637.60$$

Example 4:

Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

This problem is best described with a sketch. Here is what we're looking for.



$x^2+y^2=r^2$

$\boxed{x^2+y^2=16}$

$y=\sqrt{16-x^2}$

$A=(2x)(2y)$

$A=4xy$

$A=4x(16-x^2)^{1/2}$

$A'=4(16-x^2)^{1/2}+4x\left[\frac{1}{2}(16-x^2)^{-1/2}(-2x)\right]$

$0=4(16-x^2)^{1/2}-4x^2(16-x^2)^{-1/2}$

Factor:

$0=4(16-x^2)^{-1/2}\left[(16-x^2)^1-x^2\right]$

m^2-m^5

$m^2(m-m^3)$

lowest

$0=\dfrac{4(16-2x^2)}{(16-x^2)^{1/2}}$

Critical Values:

$16-2x^2=0$
$16=2x^2$
$8=x^2$
$\pm\sqrt{8}=x$

$16-x^2=0$
$16=x^2$
$\cancel{\pm 4=x}$

Solution:

$x=\sqrt{8}$

$y=\sqrt{16-(\sqrt{8})^2}$

$y=\sqrt{8}$

$2\sqrt{8}$ by $2\sqrt{8}$

$4\sqrt{2} \times 4\sqrt{2}$

$$0 = 4(16-x^2)^{1/2} - 4x^2(16-x^2)^{-1/2}$$

$\sqrt{16-x^2}$ $\sqrt{16-x^2}$

$$0 = 4\sqrt{16-x^2} - \frac{4x^2}{\cancel{\sqrt{16-x^2}}} \quad \sqrt{16-x^2}$$

$$0 = 4(16-x^2) - 4x^2$$

$$0 = 64 - 4x^2 - 4x^2$$

$$8x^2 = 64$$

$$x^2 = 8$$

$$x = \sqrt{8}$$