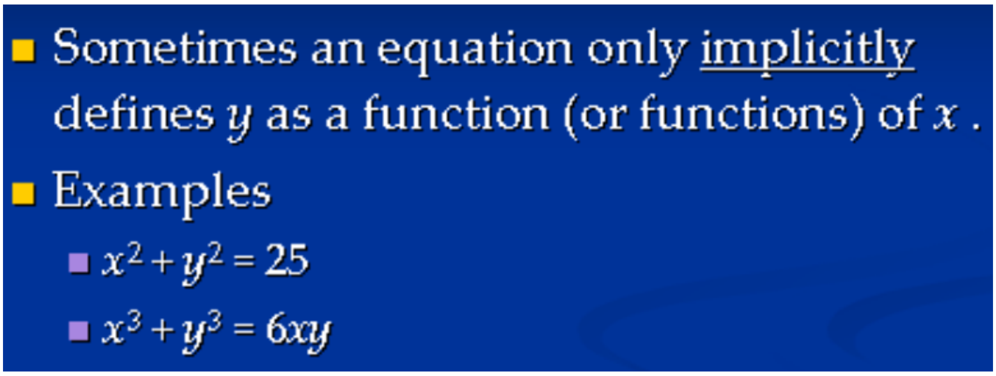
Implicit Differentiation

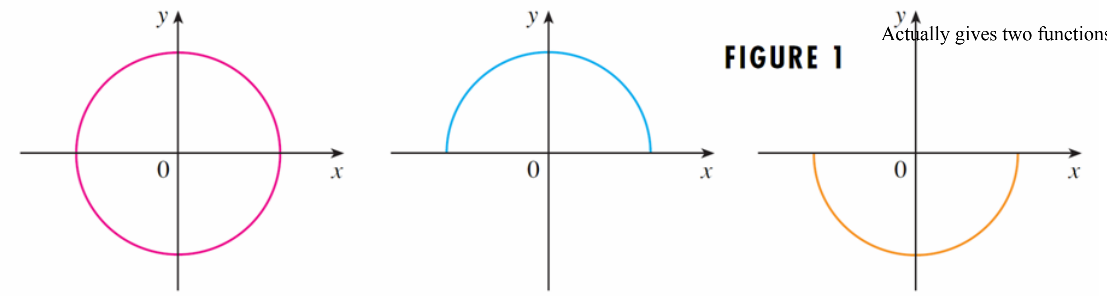


- The first equation could easily be rearranged for $y =$...

$$y = \pm\sqrt{25 - x^2}$$

Actually gives two functions

FIGURE 1



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

- The other sample equation

$$x^3 + y^3 = 6xy$$

  - can be solved for y but
  - the results are very complicated.

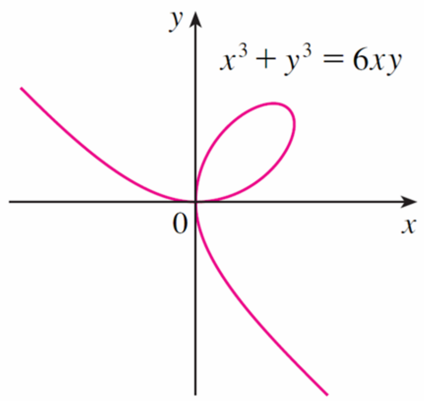


FIGURE 2 The folium of Descartes

$y = x^2$

$\frac{dy}{dx} = 2x$

$y = (x)^2$

$y' = 2(x)^1 (1)$

$\frac{dy}{dx}$

$(y)^2 = x^2$

$2(y)^1 \frac{dy}{dx}$

$(y)^7$

$7(y)^6 \frac{dy}{dx}$

Implicit Differentiation

- There is a way called *implicit differentiation* to find dy/dx without solving for y :
  - First differentiate both sides of the equation with respect to x ;
  - Then solve the resulting equation for y' .
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
  a) dy/dx
  b) an equation of the tangent at the point (3, 4) .

Solution:

Start by differentiating both sides of the equation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus... $2x + 2y\frac{dy}{dx} = 0$

Solving for $\frac{dy}{dx}$ ... $\frac{dy}{dx} = -\frac{x}{y}$

Therefore at the point (3,4) the equation of the tangent would be...

$$y - 4 = -\tfrac{3}{4}(x - 3) \qquad \text{or} \qquad 3x + 4y = 25$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate f:

$$f'(x) = \tfrac{1}{2}(25 - x^2)^{-1/2}\frac{d}{dx}(25 - x^2)$$

$$= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}$$

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

Solution (cont'd)

- So $f'(3) = -\frac{3}{\sqrt{25-3^2}} = -\frac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.
- Note that although this problem could be done both ways, implicit differentiation was easier!

Example

- For the folium of Descartes $x^3 + y^3 = 6xy$,
  - Find y' ($\frac{dy}{dx}$)
  - Find the tangent to the curve at the point (3, 3)
  - At what points on the curve is the tangent line horizontal?

$$x^3 + y^3 = 6xy$$

Product rule

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{\frac{dy}{dx}(3y^2 - 6x)}{3y^2 - 6x} = \frac{6y - 3x^2}{3y^2 - 6x}$$