Integrals Involving Trigonometry

$$\int \sin x \cos^3 x\, dx = \int \sin x \overset{\cos^2 x}{(1 - \sin^2 x)} \cos x\, dx = \int (u - u^3) du = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C$$

$$-\int (\cos x)^3 (-\sin x)\, dx$$

$$= -\frac{1}{4}\cos^4 x + C$$

$u = \sin x$

$du = \cos x\, dx$

$$\int u(1-u^2)\,du$$

$$\int (u-u^3)\,du$$

$$= \left(\frac{u^2}{2} - \frac{u^4}{4}\right)$$

$$= \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C$$

Basic Trig. Identities

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

Quotient

$$\frac{\sin x}{\cos x} = \tan x$$

$$\frac{\cos x}{\sin x} = \cot x$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

These identities look familiar??

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}, \quad \sin x \cos x = \frac{1}{2}\sin 2x$$

They might help with one like this...

$$\int \sin x \cos^3 x \, dx = \int \frac{1}{2}\sin 2x \frac{1+\cos 2x}{2} dx = \int \left(\frac{\sin 2x}{4} + \frac{\sin 4x}{8}\right) dx = -\frac{\cos 2x}{8} - \frac{\cos 4x}{16} + C.$$

$$\int \sin x \cos^2 x \cos x \, dx$$

$$\cos 2\cancel{x} = 2\cos^2 x - 1$$

$$\sin 2x = 2\sin x \cos x$$

$$\sin x \cos x = \frac{\sin 2x}{2}$$

$$\int \left(\frac{1}{2}\sin 2x\right)\left(\frac{\cos 2x + 1}{2}\right) dx$$

$$\frac{\cos 2x + 1}{2} = \cos^2 x$$

$$\frac{\sin 4x}{2}$$

$$\int \left(\frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}\sin 2x\right) dx$$

$$\int \left(\frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x\right) dx$$

$$\boxed{\sin 4x = 2\sin(2x)\cos(2x)}$$

$$\frac{\sin 4x}{2} = \sin 2x \cos(2x)$$

Integration using Trigonometric Substitution

- Method of integration used to evaluate integrals involving...

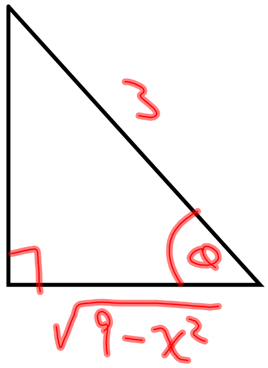
$$\sqrt{x^2 - a^2} \qquad \sqrt{x^2 + a^2} \qquad \sqrt{a^2 - x^2} \qquad a \in \mathbb{R}$$

Example:

Represent $\sqrt{9-x^2}$ using a trigonometric ratio.

$\sin\theta = \frac{x}{3}$ $\qquad$ $\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$ $\qquad$ $\theta = \cos^{-1}\left(\frac{\sqrt{9-x^2}}{3}\right)$



- Express θ as an inverse trigonometric ratio

Example: $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$

Easy!!

$\sin\theta = \frac{x}{3}$

$3\sin\theta = x$

$3\cos\theta\, d\theta = dx$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

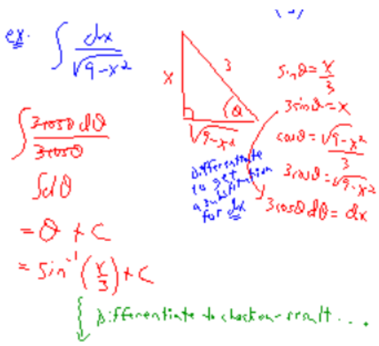
$3\cos\theta = \sqrt{9-x^2}$

$\displaystyle\int \frac{3\cos\theta\, d\theta}{3\cos\theta}$

$\displaystyle\int d\theta$

$= \theta + C$ Now Re-Substitute

$= \sin^{-1}\left(\frac{x}{3}\right) + C$



$$\int \frac{dx}{\sqrt{9-x^2}} \qquad d(\sin^{-1} u) = \frac{du}{\sqrt{1-u^2}}$$

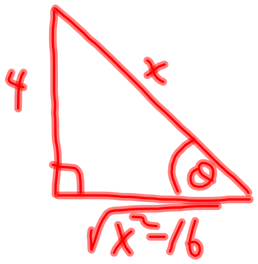
$$\int \frac{dx}{\sqrt{9\left(1-\frac{1}{9}x^2\right)}} \qquad \frac{1}{\sqrt{ab}} = \frac{1}{\sqrt{a}\sqrt{b}}$$

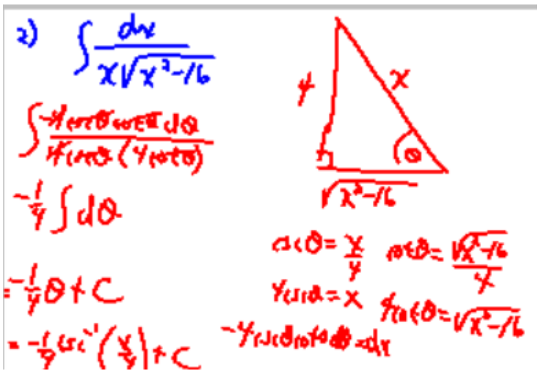
$$\cancel{(3)}\,\frac{1}{\cancel{3}}\int \frac{dx\left(\frac{1}{3}\right)}{\sqrt{1-\left(\frac{1}{3}x\right)^2}}$$

$$= \sin^{-1}\left(\frac{1}{3}x\right) + C$$

Example:

$$\int \frac{dx}{x\sqrt{x^2-16}}$$





$\csc\theta = \frac{x}{4}$

$4\csc\theta = x$

$-4\csc\theta\cot\theta\, d\theta = dx$

$\cot\theta = \frac{\sqrt{x^2-16}}{4}$

$4\cot\theta = \sqrt{x^2-16}$

$$\int \frac{-\cancel{4}\,\cancel{\csc\theta}\,\cancel{\cot\theta}\,d\theta}{4\,\cancel{\csc\theta}\,\cancel{4}\,\cancel{\cot\theta}}$$

$$-\frac{1}{4}\int d\theta$$

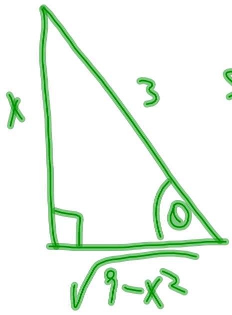
$$= -\frac{1}{4}\theta + C$$

$$= -\frac{1}{4}\csc^{-1}\frac{x}{4} + C$$

Example:

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}$$

Hint: Will require trigonometric identities



$\sin\theta = \frac{x}{3}$

$3\sin\theta = x$

$3\cos\theta\, d\theta = dx$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$3\cos\theta = \sqrt{9-x^2}$

$$\int \frac{(3\sin\theta)^2\,\cancel{3\cos\theta}\,d\theta}{\cancel{3\cos\theta}}$$

3) $\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

$\int \frac{(9\sin^2\theta)(\cancel{3\cos\theta})\,d\theta}{\cancel{3\cos\theta}}$

$\sin\theta = \frac{x}{3}$ $\quad \cos\theta = \frac{\sqrt{9-x^2}}{3}$

$3\sin\theta = x$ $\quad 3\cos\theta = \sqrt{9-x^2}$

$3\cos\theta\, d\theta = dx$

$9\int \sin^2\theta\, d\theta$

use Identity

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\boxed{\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)}$

$= 9\int \frac{1}{2}(1-\cos 2\theta)\,d\theta$

$= \frac{9}{2}\int (1-\cos 2\theta)\,d\theta$

$= \frac{9}{2}\left[\int 1\,d\theta - \frac{1}{2}\int \cos 2\theta\,(2)\,d\theta\right]$

$\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = 1 + \cos 2\theta$

$\boxed{\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)}$

$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$

$= \frac{9}{2}\left(\theta - \frac{1}{\cancel{2}}(\cancel{2}\sin\theta\cos\theta)\right) + C$

$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right) + C$

$$9\int \sin^2\theta\, d\theta$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\frac{1-\cos 2\theta}{2} = \sin^2\theta$

$$9\int\left(\frac{1}{2} - \frac{\cos 2\theta}{2}\right)d\theta$$

$$\frac{9}{2}\int d\theta - \frac{9}{2}\left(\frac{1}{2}\right)\int \cos 2\theta\,(2)\,d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C$$

$2(\sin\theta)\cos\theta$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{4}(2)\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$