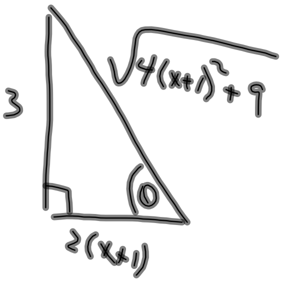
$$\int \frac{dx}{\left(4x^2+8x+13\right)^2}$$ ⟵ This will require us to complete the square on the denominator in order to set up a trigonometric substitution

$$\left[4\left(x^2+2x+1\right)+13-4\right]^2$$

$$\left[4(x+1)^2+9\right]^2 \Longrightarrow$$



$$\cot\theta=\frac{2(x+1)}{3}$$

$$3\cot\theta=2(x+1) \qquad 3\csc\theta=\sqrt{4(x+1)^2+9}$$

$$-3\csc^2\theta\, d\theta=2\,dx \qquad 9\csc^2\theta=4(x+1)^2+9$$

$$-\frac{3}{2}\csc^2\theta\, d\theta=dx$$

$$-\frac{3}{2}\int\frac{\csc^2\theta\, d\theta}{\left(9\csc^2\theta\right)^2}$$

$$-\frac{\cancel{3}}{2(\cancel{81})_{27}}\int\frac{\csc^2\theta\, d\theta}{\csc^4\theta}$$

$$-\frac{1}{54}\int\frac{1}{\csc^2\theta}\, d\theta \qquad \int\sin^2\theta\, d\theta=??$$

$$-\frac{1}{54}\int\sin^2\theta\, d\theta \Longrightarrow$$ <u>Double Angle Identity</u>

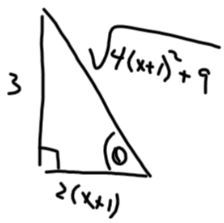
$$\cos 2\theta=1-2\sin^2\theta$$

$$\sin^2\theta=\frac{1-\cos 2\theta}{2}$$

$$-\frac{1}{54}\left[\frac{1}{2}\int d\theta-\frac{1}{(2)2}\int\cos 2\theta\,(2)\,d\theta\right]$$

$$-\frac{1}{108}\theta-\frac{1}{216}\sin 2\theta+C$$

$$-\frac{1}{108}\theta-\frac{1}{216}\left(2\sin\theta\cos\theta\right)+C$$



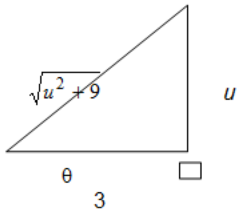
$$=-\frac{1}{108}\cot^{-1}\left(\frac{2(x+1)}{3}\right)-\frac{1}{108}\left(\frac{3}{\sqrt{4(x+1)^2+9}}\right)\left(\frac{2(x+1)}{\sqrt{4(x+1)^2+9}}\right)+C$$

Solution:

$$\int \frac{dx}{\left(4x^2+8x+13\right)^2} = \int \frac{dx}{\left(4\left(x^2+2x+1\right)+13-4\right)^2}$$

$$= \int \frac{dx}{\left(4\left(x+1\right)^2+9\right)^2} \qquad u^2 = 4\left(x+1\right)^2 \quad u = 2\left(x+1\right) \quad du = 2\,dx$$

$$= \frac{1}{2}\int \frac{du}{\left(u^2+9\right)^2}$$



$$\tan\theta = \frac{u}{3}, \qquad 3\tan\theta = u, \qquad 3\sec^2\theta\, d\theta = du,$$

$$\frac{\sqrt{u^2+9}}{3} = \sec\theta, \ \sqrt{u^2+9} = 3\sec\theta$$

$$\int \frac{dx}{\left(4x^2+8x+13\right)^2} = \frac{1}{2}\int \frac{du}{\left(u^2+9\right)^2}$$

$$= \frac{1}{2}\int \frac{3\sec^2\theta\, d\theta}{\left(3\sec\theta\right)^4}$$

$$= \frac{1}{2}\int \frac{3\sec^2\theta\, d\theta}{81\sec^4\theta}$$

$$= \frac{1}{54}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{54}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{54}\int \frac{(1+\cos 2\theta)d\theta}{2}$$

$$= \frac{1}{108}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{108}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{u}{3} + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{u}{3} + \frac{u}{\sqrt{u^2+9}}\,\frac{3}{\sqrt{u^2+9}}\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{u}{3} + \frac{3u}{u^2+9}\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{2\left(x+1\right)}{3} + \frac{6\left(x+1\right)}{4\left(x+1\right)^2+9}\right) + C$$

Sometimes long division can be a useful strategy to evaluate an integral...

If numerator is same degree or higher than the denominator then we must first perform long division!!

$$\int \frac{x^3+4}{x+1}\,dx \qquad \frac{7}{3} = 2\tfrac{1}{3} = 2 + \frac{1}{3}$$

$$\begin{array}{r} x^2 - x + 1 \\ x+1 \overline{) x^3 + 4} \\ \underline{x^3 + x^2} \\ -x^2 + 4 \\ \underline{-x^2 - x} \\ x + 4 \\ \underline{x + 1} \\ 3 \end{array}$$

Synthetic

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & 4 \\ & & 1 & -1 & 1 \\ \hline & 1 & -1 & 1 & 3 \end{array}$$

$(x^2 - x + 1)$ Rem 3

$$\int \frac{x^3+4}{x+1}\,dx = \int \left[(x^2 - x + 1) + \frac{3}{x+1} \right] dx$$

$$\int (x^2 - x + 1)\,dx + \int \frac{3}{x+1}\,dx$$

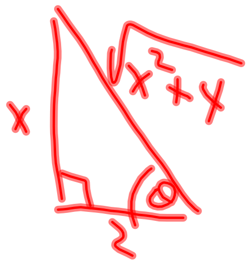
$$= \frac{x^3}{3} - \frac{x^2}{2} + x + 3\ln|x+1| + C$$

$$\int \frac{3x^3+2x+5}{x^2+4}\,dx$$

$$\begin{array}{r} 3x \\ x^2+4 \;\big)\overline{\,3x^3+2x+5\,} \\ \underline{3x^3+12x} \\ -10x+5 \end{array}$$

$$\int 3x\,dx + \int \frac{-10x+5}{x^2+4}\,dx$$

$$\int 3x\,dx - \frac{10}{(2)}\int \frac{(2)x}{x^2+4}\,dx + 5\int \frac{dx}{x^2+4}$$



$$\frac{3}{2}x^2 - 5\ln(x^2+4) + 5\int \frac{2\cancel{\sec^2\theta}\,d\theta}{4\cancel{\sec^2\theta}}$$

$$+\frac{5}{2}\int d\theta$$

$$+\frac{5}{2}\theta$$

$$\boxed{= \frac{3}{2}x^2 - 5\ln(x^2+4) + \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

$2\tan\theta = x$

$2\sec^2\theta\,d\theta = dx$

$2\sec\theta = \sqrt{x^2+4}$

$4\sec^2\theta = x^2+4$

Integration using Partial Fractions

Simplify: $\frac{3}{x-5}+\frac{2}{x+4}$

$$=\frac{3(x+4)+2(x-5)}{(x-5)(x+4)}$$

$$=\frac{5x+2}{x^2-x-20}$$

We want to reverse the process of finding a common denominator...

Express as partial fractions: $\frac{5x+2}{x^2-x-20}$

1. Factor the denominator: $\frac{5x+2}{(x-5)(x+4)}$

2. Separate into partial fractions:

$$\frac{A}{x-5}+\frac{B}{x+4}=\frac{5x+2}{x^2-x-20}$$

3. Find common denominator and solve for A and B:

$$\frac{A(x+4)+B(x-5)}{(x-5)(x+4)}=\frac{5x+2}{x^2-x-20}$$

$$\frac{Ax+4A+Bx-5B}{x^2-x-20}=\frac{5x+2}{x^2-x-20}$$

$$\therefore Ax+Bx+4A-5B=5x+2$$

$A+B=5$

$A=5-B$

$4A-5B=2$

$4(5-B)-5B=2$

$20-4B-5B=2$

$-9B=-18$

$B=2$

$A=5-2$

$A=3$

$$\frac{3}{x-5}+\frac{2}{x+4}=\frac{5x+2}{x^2-x-20}$$

Now let's evaluate the following integral...

(See Prior Page)

$$\int \frac{(5x+2)dx}{x^2 - x - 20}$$

$$\int \left(\frac{3}{x-5} + \frac{2}{x+4} \right) dx$$

$$3\int \frac{dx}{x-5} + 2\int \frac{dx}{x+4}$$

$$= 3\ln|x-5| + 2\ln|x+4| + C$$

Here is another example...

$$\int \frac{x\,dx}{x^2 - 3x + 2}$$

$$\int \frac{x\,dx}{(x-2)(x-1)} \Rightarrow \frac{A}{x-2} + \frac{B}{x-1} = \frac{x}{x^2-3x+2}$$

$$A(x-1) + B(x-2) = x$$

$$Ax + Bx - A - 2B = x$$

$$A + B = 1 \qquad -A - 2B = 0$$

$$-2B + B = 1 \qquad -A = 2B$$

$$-B = 1 \qquad A = -2B$$

$$B = -1 \qquad A = 2$$

$$2\int \frac{dx}{x-2} - \int \frac{dx}{x-1}$$

$$\boxed{2\ln|x-2| - \ln|x-1| + C}$$

Some special situations involving partial fractions...

Note 1:

- If the degree of the numerator is the same as that of the denominator, or higher, we would have to take the preliminary step of first performing a long division.

Ex. $$\frac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1}$$

$$2x^2 + x + 1 \overline{\big)\, 2x^3 - 11x^2 - 2x + 2} = (x-6) + \frac{5x-4}{(x+1)(2x-1)}$$

Note 2:

- If the denominator has more than two linear factors, we must include a term corresponding to each factor.

Ex. $$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}$$

Note 3:

- If a linear factor is repeated, we need to include extra terms in the partial fraction expression.

Ex. $$\frac{x}{(x+3)^2(x-2)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-2}$$

$$\frac{3}{(x+5)^2} - \frac{2}{x+5}$$

$$\frac{}{(x+5)^2}$$

Note 4:

- When we factor the denominator as far as possible, it may happen that we end up with an irreducible quadratic factor of the form $ax^2 + bx + c$, where the discriminant is negative. Then the corresponding partial fraction is of the form...

$$\frac{Ax + B}{ax^2 + bx + c}$$

where A and B are constants to be determined. This term can be integrated by completing the square and by using the integration formula...

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$ ***(or a trig. substitution)***