Some special situations involving partial fractions...

Note 1:

- If the degree of the numerator is the same as that of the denominator, or higher, we would have to take the preliminary step of first performing a long division.

Ex. $$\frac{2x^3 - 11x^2 - 2x + 2}{2x^2 + x - 1}$$

$$2x^2 + x + 1 \overline{\smash{)}\,2x^3 - 11x^2 - 2x + 2} = (x-6) + \frac{5x-4}{(x+1)(2x-1)}$$

Note 2:

- If the denominator has more than two linear factors, we must include a term corresponding to each factor.

Ex. $$\frac{x+6}{x(x-3)(4x+5)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{4x+5}$$

Note 3:

- If a linear factor is repeated, we need to include extra terms in the partial fraction expression.

Ex. $$\frac{x}{(x+3)^2(x-2)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x-2}$$

$x^2 - 7 = 0$ $x^2 + 3 = 0$
$D < 0$

Note 4:

- When we factor the denominator as far as possible, it may happen that we end up with an irreducible quadratic factor of the form $ax^2 + bx + c$, where the discriminant is negative. Then the corresponding partial fraction is of the form...

$$\frac{Ax + B}{ax^2 + bx + c} \leftarrow \text{Degree Less}$$

where A and B are constants to be determined. This term can be integrated by completing the square and by using the integration formula...

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$ ***(or a trig. substitution)***

Examples...

(a) $\displaystyle\int \frac{x-1}{x(x^2+2x+1)}\,dx$

$$\int \frac{x-1}{x(x+1)^2}dx$$

$$\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}=\frac{x-1}{x(x+1)^2}$$

$$\frac{A(x+1)^2+B(x+1)(x)+Cx}{x(x+1)^2}=\frac{x-1}{x(x+1)^2}$$

$$A(x^2+2x+1)+Bx(x+1)+Cx=x-1$$

$$Ax^2+Bx^2+2Ax+Bx+Cx+A=x-1$$

$A+B=0$ $\quad 2A+B+C=1$ $\quad \boxed{A=-1}$

$-1+B=0$

$\boxed{B=1}$

$2(-1)+1+C=1$

$\boxed{C=2}$

$$-1\int\frac{1}{x}dx+\int\frac{dx}{x+1}+2\int\frac{dx}{(x+1)^2} \quad 2\int(x+1)^{-2}dx$$

$$=-\ln|x|+\ln|x+1|-2(x+1)^{-1}+C$$

(b) $\displaystyle\int \frac{2x^2+x+16}{(x-2)(x^2+9)}\,dx$

$$\frac{A}{x-2}+\frac{Bx+C}{x^2+9}=\frac{2x^2+x+16}{(x-2)(x^2+9)}$$

$$A(x^2+9)+(Bx+C)(x-2)=2x^2+x+16$$

$$Ax^2+9A+Bx^2-2Bx+Cx-2C=2x^2+x+16$$

$A+B=2$

$B=2-A$

$-2B+C=1$

$9A-2C=16$

$-2(2-A)+C=1$

$-4+2A+C=1$

$2A+C=5$

$C=5-2A$

$B=2-2$

$\underline{B=0}$

$9A-2(5-2A)=16$

$9A-10+4A=16$

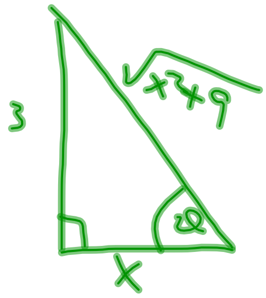
$13A=26$

$A=2$

$C=5-2(2)$

$C=1$

$$2\int\frac{dx}{x-2}+\int\frac{1\,dx}{x^2+9} \Longrightarrow$$



$$-\int\frac{3\cancel{\csc^2\theta}\,d\theta}{\cancel{\csc^2\theta}}$$

$=-3\theta$

$=-3\cot^{-1}\left(\frac{x}{3}\right)$

$3\cot\theta=x$

$-3\csc^2\theta\,d\theta=dx$

$\csc\theta=\sqrt{x^2+9}$

$\csc^2\theta=x^2+9$

$$\boxed{2\ln|x-2|-3\cot^{-1}\left(\frac{x}{3}\right)+C}$$

Practice Questions...

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#9, 10, 11, 13, 14, 17, 19, 21, 23, 25, 27

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Let's borrow a few examples from final exams given to the Wildcats over the past years...

Evaluate $\displaystyle\int \frac{(x+2)(x+4)}{(x+1)(x+3)(x+5)}\,dx$.

$$\int \frac{x^3+2x-1}{x^2-1}\,dx$$

Evaluate $\displaystyle\int \frac{dx}{x^2+6x+13}$

Evaluate $\displaystyle\int_0^1 2^{-x}\,dx$.

$$\int \frac{\sqrt{x^2-1}}{x}\,dx$$

Evaluate $\displaystyle\int \frac{\cos^5(x)}{\sin^2(x)}\,dx$.

Evaluate $\int \frac{(x+2)(x+4)}{(x+1)(x+3)(x+5)}\,dx$.

$$\frac{A}{x+1}+\frac{B}{x+3}+\frac{C}{x+5} = \frac{x^2+6x+8}{(x+1)(x+3)(x+5)}$$

$$A(x+3)(x+5)+B(x+1)(x+5)+C(x+1)(x+3)=x^2+6x+8$$

$$A(x^2+8x+15)+B(x^2+6x+5)+C(x^2+4x+3)=x^2+6x+8$$

$$A+B+C=1 \qquad 8A+6B+4C=6 \qquad 15A+5B+3C=8$$

$$A=1-B-C$$

$$8(1-B-C)+6B+4C=6 \qquad 15(1-B-C)+5B+3C=8$$

$$8-2B-4C=6 \qquad 15-10B-12C=8$$

$$-2B-4C=-2 \qquad \boxed{7=10B+12C}$$

$$\boxed{2B+4C=2}$$

$$6B+12C=6$$
$$10B+12C=7$$
$$4B=1$$
$$B=\frac{1}{4}$$

$$2\left(\frac{1}{4}\right)+4C=2$$

$$4C=2-\frac{1}{2}$$

$$4C=\frac{3}{2}$$

$$C=\frac{3}{8}$$

$$A=1-\frac{1}{4}-\frac{3}{8}$$

$$A=\frac{8-2-3}{8}=\frac{3}{8}$$

$$\frac{3}{8}\int\frac{dx}{x+1}+\frac{1}{4}\int\frac{dx}{x+3}+\frac{3}{8}\int\frac{dx}{x+5}$$

$$\frac{3}{8}\ln|x+1|+\frac{1}{4}\ln|x+3|+3 \ldots$$

$$\int \frac{x^3+2x-1}{x^2-1}\,dx = \int x\,dx + \int \frac{3x-1}{x^2-1}\,dx$$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3+2x-1 \\ & x^3-x \\ \hline & 3x-1 \end{array}$$

$$\int \frac{3x-1}{(x-1)(x+1)}\,dx$$

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{3x-1}{(x-1)(x+1)}$$

$$\int x\,dx + \int \frac{dx}{x-1} + 2\int \frac{dx}{x+1}$$

$$A(x+1) + B(x-1) = 3x-1$$

$$= \frac{x^2}{2} + \ln|x-1| + 2\ln|x+1| + C$$

$A+B=3 \quad A-B=-1$

$$\begin{array}{r} A-B=-1 \\ \hline 2A=2 \end{array}$$

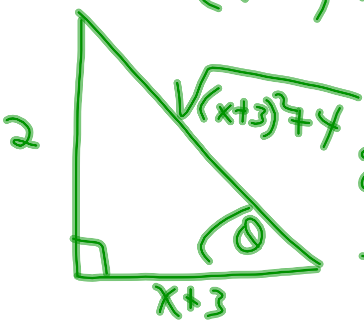
$A=1 \qquad 1-B=-1$

$2=B$

Evaluate $\int \frac{dx}{x^2+6x+13} \Longrightarrow (x^2+6x+9)-9+13$

$$(x+3)^2+4$$

$$\int \frac{dx}{(x+3)^2+4}$$



$$2\cot\theta = x+3$$

$$-2\csc^2\theta\, d\theta = dx$$

$$(2\csc\theta)^2 = \left(\sqrt{(x+3)^2+4}\right)^2$$

$$4\csc^2\theta = (x+3)^2+4$$

$$\int \frac{-2\csc^2\theta\, d\theta}{4\csc^2\theta}$$

$$-\frac{1}{2}\int d\theta$$

$$-\frac{1}{2}\theta + C$$

$$-\frac{1}{2}\cot^{-1}\left(\frac{x+3}{2}\right)+C$$

Evaluate $\int_0^1 2^{-x}\,dx$.

$b^u \ln b \, du$

$$\frac{1}{-\ln 2}\int_0^1 2^{-x}(\ln 2)(-1)\,dx$$

$$-\frac{1}{\ln 2}\,2^{-x}\Big|_0^1$$

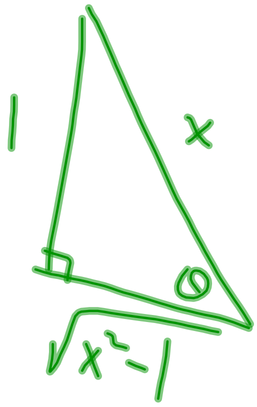
$$= \frac{-1}{\ln 2}\left(2^{-1} - 2^{0}\right)$$

$$= \frac{-1}{\ln 2}\left(\frac{1}{2} - \frac{2}{2}\right)$$

$$= \frac{1}{2\ln 2} = \frac{1}{\ln 4}$$

2^2

$$\int \frac{\sqrt{x^2-1}}{x}\,dx$$



$\cot\theta = \sqrt{x^2-1}$

$\csc\theta = x$

$-\csc\theta\cot\theta\,d\theta = dx$

$$\int \frac{\cot\theta\,(-\cancel{\csc\theta}\cot\theta)\,d\theta}{\cancel{\csc\theta}}$$

$$\int -\cot^2\theta\,d\theta$$

$1+\cot^2\theta = \csc^2\theta$

$1-\csc^2\theta = -\cot^2\theta$

$$\int 1-\csc^2\theta\,d\theta$$

$$\int d\theta - \int \csc^2\theta\,d\theta$$

$$= \theta + \cot\theta + C$$

$$= \csc^{-1}x + \sqrt{x^2-1} + C$$

Evaluate $\int \frac{\cos^5(x)}{\sin^2(x)}\,dx$.

$$\int \frac{(1-\sin^2 x)^2 \cos x\, dx}{\sin^2 x}$$

$$\int \frac{(1-2\sin^2 x+\sin^4 x)}{\sin^2 x}\cos x\, dx$$

$$\int \left[(\sin x)^{-2} - 2 + (\sin x)^2\right]\cos x\, dx$$

$$\int (\sin x)^{-2}\cos x\, dx - 2\int \cos x\, dx + \int (\sin x)^2 \cos x\, dx$$

$$\boxed{= -(\sin x)^{-1} - 2\sin x + \frac{1}{3}\sin^3 x + C}$$

Set A

2. Evaluate the following integrals.

(3) (a) $\displaystyle\int \frac{x^2}{\sqrt{x-4}}dx$

(4) (b) $\displaystyle\int \frac{x\arcsin(x^2)}{\sqrt{1-x^4}}dx$

(4) (c) $\displaystyle\int_0^{\pi/4} \sqrt{\tan x}\sec^4 x dx$

(5) (d) $\displaystyle\int (\cos^2\theta + \sin^3\theta)d\theta$

(4) (e) $\displaystyle\int \frac{\sqrt{9x^2-4}}{x}dx$

(4) (f) $\displaystyle\int \frac{6x^2-5x-1}{(x-2)(x^2+9)}dx$

(4) (g) $\displaystyle\int 16x(\arctan(4x))dx$

Set B

(5) (a) $\displaystyle\int_1^5 \frac{x+2}{\sqrt{2x-1}}\,dx$

(5) (b) $\displaystyle\int \frac{1}{x^3\sqrt{x^2-4}}\,dx$

(5) (c) $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

(5) (d) $\displaystyle\int \frac{\sec^4\sqrt{x}\,\tan^2\sqrt{x}}{\sqrt{x}}\,dx$

(5) (e) $\displaystyle\int_0^{\frac{1}{2}} \frac{x+\arccos x}{\sqrt{1-x^2}}\,dx$

(5) (f) $\displaystyle\int \frac{e^x}{\sqrt{3-2e^x-e^{2x}}}\,dx$

(5) (g) $\displaystyle\int \frac{3x^2-2}{x^2-2x-8}\,dx$

Attachments

Worksheet - Finding the Equation.doc