Exponential Functions

Did You Know?

Radium was once an additive in toothpaste, hair creams, and even food items due to its supposed curative powers. Once it was discovered that radium is over one million times as radioactive as the same mass of uranium, these products were prohibited because of their serious adverse health effects.

Key Terms

exponential function
exponential growth
half-life
exponential equation

Exponential Functions...

$b>0$, $b \neq 1$

$\boxed{y = b^x}$ — Base (common ratio)

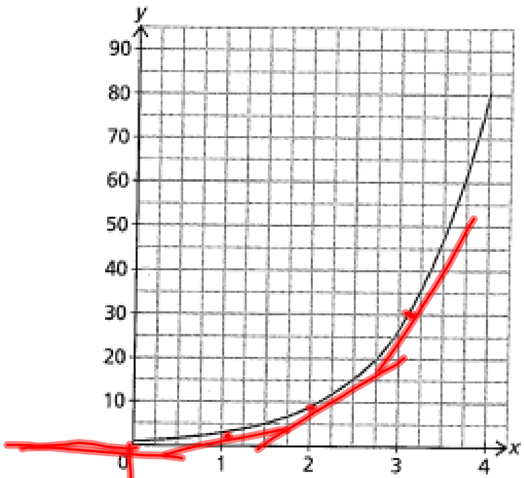
- Exponential Functions are either growth or decay curves

Step A

$y = 3^x$

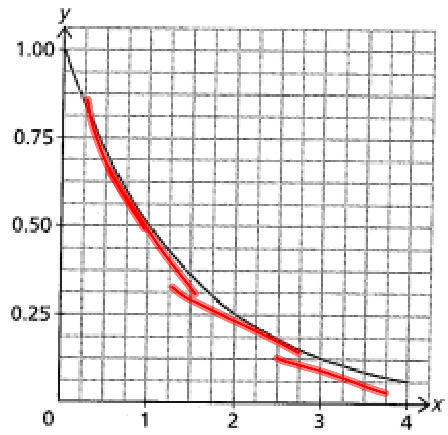
x	0	1	2	3	4
y	1	3	9	27	81

+3



$y = (0.5)^x$

x	0	1	2	3	4
y	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$



Growth $b > 1$

Ex: bacteria cultures
profit from investments

Decay $0 < b < 1$

Ex: depreciation
radioactive decay

OTHER PROPERTIES:

- The Slopes of the tangent lines are changing along the curve

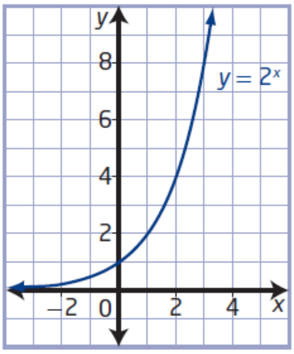
- There is a common ratio between successive y-values when the x-values change by the same increment.
  (Base of the function)

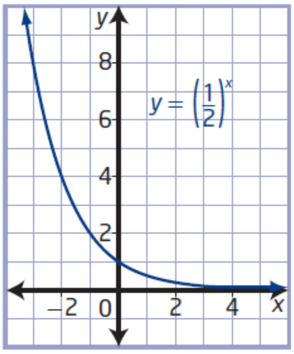
- The functions do not intersect the x-axis.
  (Horizontal Asymptote)

- They have the point (0,1) in common.
  (Initial Point)

Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $c = 1$
  - has a domain of $\{x \mid x \in R\}$
  - has a range of $\{y \mid y > 0, y \in R\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$





Follow Up...

Determine the common ratio for each of the following:

X	Y1
1	-3
2	-4.5
3	-6.75
4	-10.13

Common Ratio $\Rightarrow$ 1.5

$= \frac{-4.5}{-3} = 1.5$

$y = 1.5^x$

X	Y1
1	.6
2	.12
3	.024
4	.0048

$r = 0.2$

$y = (0.2)^x$

$$y = 2^{-x}$$

$$y = (2^{-1})^x$$

$$y = \left(\frac{1}{2}\right)^x$$

$$y = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$$

$$y = \left(\frac{1}{2}\right)^{-x}$$

$$y = \left((2^{-1})^{-1}\right)^x$$

$$y = 2^x$$

$$y = \left(\frac{1}{2}\right)^{-x}$$

$$y = \left(\frac{2}{1}\right)^x$$

$$y = 2^x$$

Transformations of the Exponential Function

$$y = \underbrace{a}_{\text{intial value}}\underbrace{(b)}_{\text{base}}{}^{x}$$

check with...

Properties:

If $b > 1$, then the graph will be GROWTH

If $0 < b < 1$, then the graph will be DECAY

y - intercept: happens when $x = 0$, so... $\boxed{y\text{-int} = a}$

Transformations of the Exponential Function

$$y = a(b)^x + d$$

check with...

intial value

base

vertical translation

Properties:

If $b > 1$, then the graph will be GROWTH

If $0 < b < 1$, then the graph will be DECAY

y - intercept: happens when $x = 0$, so... $\boxed{y\text{-int} = a + d}$

Horizontal Asymptote - a horizontal line that a graph approaches but never intersects.

Equation of Horizontal Asymptote will be... $\boxed{y = d}$

Domain - describes all possible x-values
Range - describes all possible y-values

Thus, for exponential functions...

Domain: $\{x \in \mathrm{R}\}$

Range: $\{y > d\}$

Horizontal Asymptote

Exercise: Complete the following table...

Equation	Growth/Decay	y-intercept	Eq'n for Horizontal Asymptote
$y = 3(5)^x - 4$	G	$(0, -1)$	$y = -4$
$y = 4\left(\frac{2}{5}\right)^x + 1$	D	$(0, 5)$	$y = 1$
$y = 2^x - 2$	G	$(0, -1)$	$y = -2$
$y = \frac{3}{4}\left(\frac{1}{2}\right)^x$	D	$(0, \frac{3}{4})$	$y = 0$
$y = 5(3)^x$	G	$(0, 5)$	$y = 0$

Check Up!!

Domain & Range??

Determine the y-intercept, the equation of the horizontal asymptote and state whether the function grows or decays:

1. $3y - 5 = 9\left(5^x\right) + 2$

$3y = 9(5^x) + 7$

$y = 3(5)^x + \frac{7}{3}$

y-Int:

$y = 3(1) + \frac{7}{3}$

$y = \frac{16}{3}$ $\left(0, \frac{16}{3}\right)$

H. Asym

$y = \frac{7}{3}$

- Growth
- D: $x \in R$
- R: $y > \frac{7}{3}$

2. $(\cancel{3})\ \frac{2}{\cancel{3}}(y - 1) \overset{(3)}{=} \overset{(3)}{6} - \left(\frac{3}{2}\right)^x$

$2y - 2 = 18 - 3\left(\frac{3}{2}\right)^x$

$2y = -3\left(\frac{3}{2}\right)^x + 20$

$y = -\frac{3}{2}\left(\frac{3}{2}\right)^x + 10$

y-Int:

$y = -\frac{3}{2} + 10$

$y = \frac{17}{2}$ $\left(0, \frac{17}{2}\right)$

H. Asym.

$y = 10$

- Growth

D: $x \in R$

R: $y > 10$

Graphing Exponential Functions

- Using Transformations:

Example:

$$\frac{1}{3}(y-3)=2^{x+1}$$

$$y=3(2)^{x+1}+3$$

$y=3(x+1)^2+3$

$y=3\sin(x+1)+3$

What transformations have been applied to $y=2^x$?

- Vertical Stretch: 3
- Vertical Translation: Up 3
- Horizontal Translation: Left 1

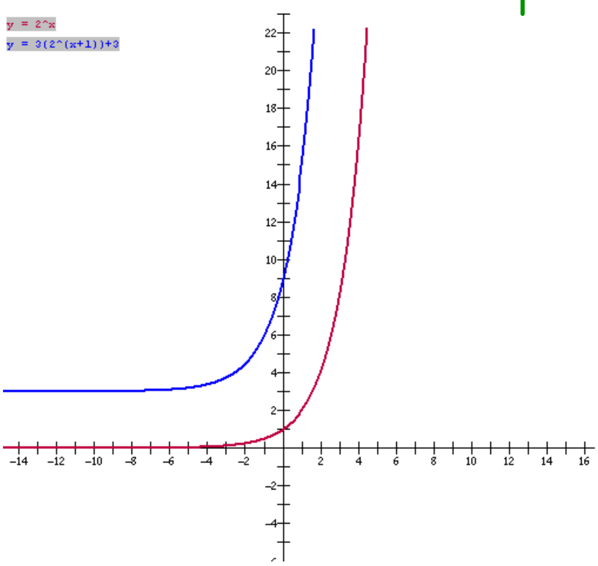
Mapping Rule: $(x,y)\rightarrow(x-1,3y+3)$

$y=2^x$

x	y
-1	1/2
0	1
1	2
2	4
3	8

$y=3(2)^{x+1}+3$

x	y
-2	9/2
-1	
0	
1	
2	



Example:

The exponential function $y = 5^x$ is transformed according to the following mapping rule:

$$(x, y) \rightarrow (x-2, 3y+6)$$

- Determine the equation of this function
- What is the y-intercept?
- What is the equation of the horizontal asymptote?
- Sketch this function

⇒ $y = 3(5^{x+2}) + 6$ or $y = 3(5)^{x+2} + 6$

⇒ $(x=0)$

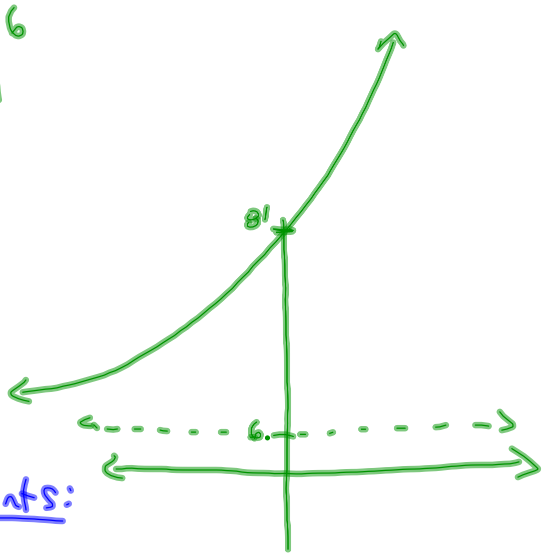
$y = 3(5^{0+2}) + 6$

$y = 75 + 6$

$y = 81$

$(0, 81)$

⇒ H. Asym.: $y = 6$



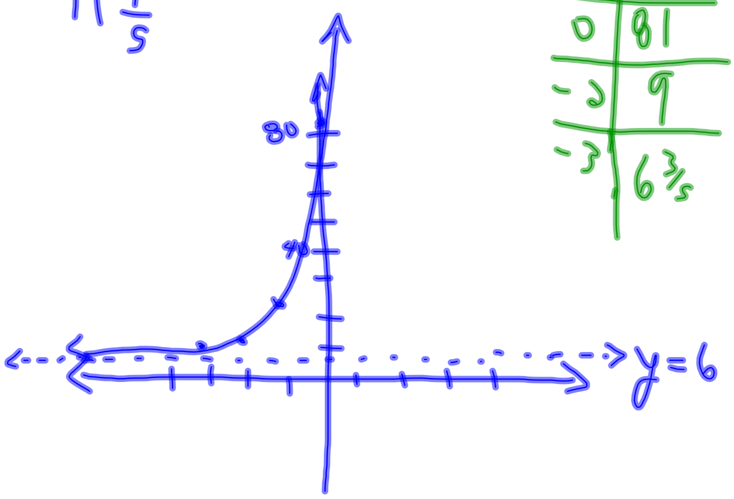
Use 4 Points:

$y = 5^x$

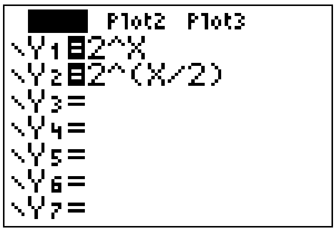
x	y
1	5
2	25
0	1
-1	$\frac{1}{5}$

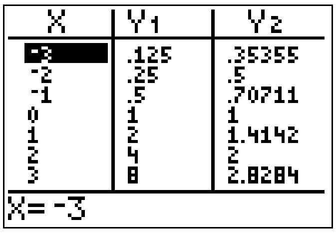
⇒ $(x-2, 3y+6)$

x	y
-1	21
0	81
-2	9
-3	$6\frac{3}{5}$



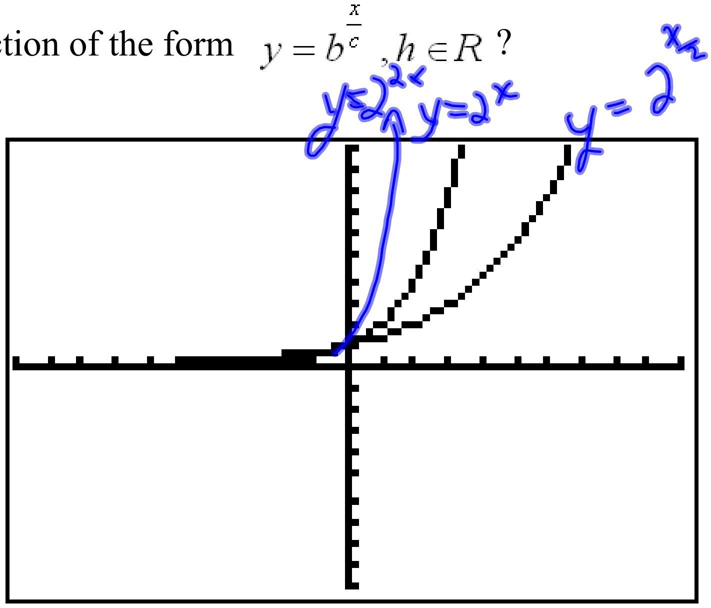
What about an exponential function of the form $y = b^{\frac{x}{c}}, h \in R$?





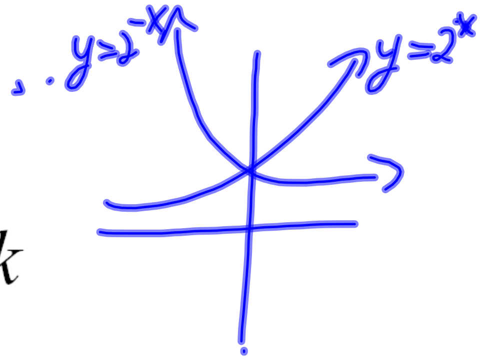
X	Y1	Y2
-3	.125	.35355
-2	.25	.5
-1	.5	.70711
0	1	1
1	2	1.4142
2	4	2
3	8	2.8284

X= -3



How is graph being stretched??

Let's Summarize...



Function Notation (Standard Form)

$$y = ab^{\frac{1}{c}(x+h)} + k$$

Mapping Notation - (with respect to $y = b^x$)

$$(x, y) \rightarrow (cx - h, ay + k)$$

where: a = vertical stretch factor

b = base (common ratio)

c = horizontal stretch factor

h = horizontal translation

k = vertical translation (or letter d can be used)

Example:

Given the exponential function shown below...

- Write a mapping that would map the graph of $y = 2^x$ to this function.
- Complete the tables of values below using the mapping rule.

$$y = 7(2)^{\frac{1}{5}(x+3)} + 4$$

$y=2^x$

$$(x, y) \rightarrow (5x-3, 7y+4)$$

maps to...

x	y
-3	1/8
-2	1/4
-1	1/2
0	1
1	2
2	4
3	8

x	y
-18	39/8
-13	23/4
-8	15/2
-3	11
2	18
7	32
12	60

Example:

Given the exponential function shown below...

- Write a mapping that would map the graph of $y = 3^x$ to this function.

$$4y + 7 = 8(3)^{5x-3} - 13$$

$$\frac{4y}{4} = \frac{8(3)^{5x-3}}{4} - \frac{20}{4}$$

$$y = 2(3)^{5(x-\frac{3}{5})} - 5$$

$$(x, y) \rightarrow \left(\frac{1}{5}x + \frac{3}{5}, 2y - 5\right)$$

$$\sin(3\theta - \pi)$$

$$3\left(\theta - \frac{\pi}{3}\right)$$

Example:

Given the mapping rule shown below is used to transform the graph of $y = 6^x$ ...

- Determine the equation of the new function.

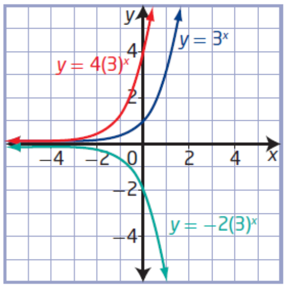
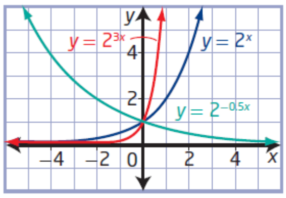
$$(x, y) \rightarrow \left(\frac{2}{5}x + 4\ ,\ \frac{3}{8}y + 6\right)$$

$$y = \frac{3}{8}(6)^{\frac{5}{2}(x-4)} + 6$$

Link the Ideas

Sec 7.2

The graph of a function of the form $f(x) = a(c)^{b(x-h)} + k$ is obtained by applying transformations to the graph of the base function $y = c^x$, where $c > 0$.

<table>
<tr><th>Parameter</th><th>Transformation</th><th>Example</th></tr>
<tr><td>a</td><td>• Vertical stretch about the x-axis by a factor of $|a|$
• For $a < 0$, reflection in the x-axis
• $(x, y) \rightarrow (x, ay)$</td><td>
</td></tr>
<tr><td>b</td><td>• Horizontal stretch about the y-axis by a factor of $\frac{1}{|b|}$
• For $b < 0$, reflection in the y-axis
• $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$</td><td>
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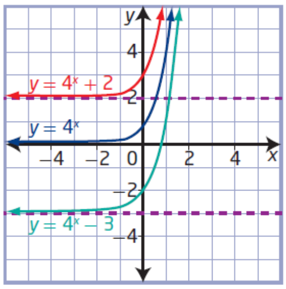
<tr><td>k</td><td>• Vertical translation up or down
• $(x, y) \rightarrow (x, y + k)$</td><td>
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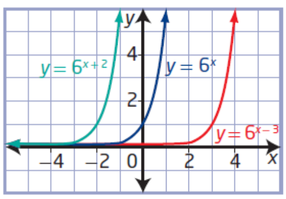
<tr><td>h</td><td>• Horizontal translation left or right
• $(x, y) \rightarrow (x + h, y)$</td><td>
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Check Up Time!!

Given the exponential function...

$y = 2^x$

$$\left(-\tfrac{1}{3}\right) -3(2y-5) \overset{\left(-\frac{1}{3}\right)}{=} -12(2)^{7x+5} + 9\left(-\tfrac{1}{3}\right)$$

- Write a mapping that would map the graph of $y = 2^x$ to this function.
- State the equation of the horizontal asymptote.
- State the range.
- State the y-intercept.
- State the coordinates of the ordered pair (3 , 8) from the base function after all of the transformations have been applied

$$2y-5 = 4(2)^{7(x+\frac{5}{7})} - 3$$

$$\frac{2}{2}y = \frac{4}{2}(2)^{7(x+\frac{5}{7})} + \frac{2}{2}$$

$$\boxed{y = 2(2)^{7(x+\frac{5}{7})} + 1}$$

1. $(x, y) \rightarrow \left(\frac{1}{7}x - \frac{5}{7}, 2y+1\right)$

2. $y = 1$

3. $y > 1$

4. $y = 2(2)^{7(0+\frac{5}{7})} + 1$

   $y = 2(2)^5 + 1$

   $y = 65$

   $(0, 65)$

5. $(3, 8) \rightarrow \left(\frac{1}{7}(3) - \frac{5}{7}, 2(8)+1\right)$

   $\rightarrow \left(-\frac{2}{7}, 17\right)$

Practice Sheets

#3, 4, 6, 7