Review: Antiderivatives

Antidifferentiate each of the following:

1. $f'(x) = 4xe^{x^2}$

$f'(x) = \frac{4}{2} e^{x^2} 2x$

$f(x) = 2e^{x^2} + C$

2. $y' = \dfrac{2}{(3x-5)^3} + \dfrac{5x}{x^2+1}$

$y' = \frac{2}{3}(3x-5)^{-3}(3) + \frac{5}{2}\left(\frac{2x}{x^2+1}\right)$

$u^n \cdot du$

$\frac{du}{u}$

$y = -\frac{1}{3}(3x-5)^{-2} + \frac{5}{2}\ln(x^2+1) + C$

Warm Up

If $\frac{dy}{dx} = \tan x$, then $y =$

$y' = \tan x$

$y' = \frac{\sin x}{\cos x} \Rightarrow y = -\ln|\cos x| + C$

(A) $\frac{1}{2}\tan^2 x + C$ (B) $\sec^2 x + C$ (C) $\ln|\sec x| + C$

(D) $\ln|\cos x| + C$ $-\left(\frac{-\sin x}{\cos x}\right)$ (E) $\sec x \tan x + C$

$y = \ln(\cos x)^{-1}$

$y = \ln\left(\frac{1}{\cos x}\right) = \ln|\sec x|$

If $f'(x) = -f(x)$ and $f(1) = 1$, then $f(x) =$

(A) $\frac{1}{2}e^{-2x+2}$ (B) e^{-x-1} (C) e^{1-x} (D) e^{-x} (E) $-e^{x}$

$e^{1-1} = e^0 = 1$

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A particle moves along the x-axis in such a way that at time $t > 0$ its position coordinate is $x = \sin(e^t)$.

(a) Find the velocity and acceleration of the particle at time t.

(b) At what time does the particle first have zero velocity?

(c) What is the acceleration of the particle at the time determined in part (b)?

a) velocity: $x' = \cos e^t (e^t)$

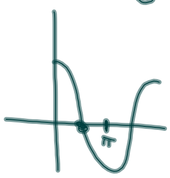
accel: $x'' = (-\sin e^t)e^t + (\cos e^t)(e^t)$

$x'' = e^t(\cos e^t - \sin e^t)$

(b) $(\cos e^t)\, e^t = 0$

$\cos e^t = 0$ OR $e^t = 0$

$\emptyset$



$e^t = \frac{\pi}{2}$

$\ln e^t = \ln\frac{\pi}{2}$

$t \ln e = \ln\frac{\pi}{2}$, $\ln e = 1$

$t = \ln\frac{\pi}{2}$

(c) $x'' = e^t(\cos e^t - \sin e^t)$

at $\ln\frac{\pi}{2}$ ...

$x'' = e^{\ln\frac{\pi}{2}}\left(\cos e^{\ln\frac{\pi}{2}} - \sin e^{\ln\frac{\pi}{2}}\right)$

$x'' = \frac{\pi}{2}\left(\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right)$

$x'' = \frac{\pi}{2}(0 - 1)$

$x'' = -\frac{\pi}{2}$

$e^{\ln\frac{\pi}{2}} = \frac{\pi}{2}$

③ $b^{\log_b M} = M$

① $\log_b 1 = 0$

② $\log_b b^M = M$

Indeterminate forms and L'Hospital's Rule

Suppose we want to examine the following limit:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{0}{0} \text{ indeterminate}$$

What is the limit of both the denominator and numerator as x approaches 2?

Determine the value of this limit.

$$\lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = x+2 = 4$$

l'hospital's Rule

$$\lim_{x \to 2} \frac{2x}{1} = \frac{2(2)}{1} = 4$$

The Indeterminate Form 0/0

- In general, a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ may or may not exist.
- Such a limit is called an *indeterminate form of type* $\frac{0}{0}$; similarly, we also consider the indeterminate form $\frac{\infty}{\infty}$.

L'Hospital's Rule

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives.

L'Hospital's Rule is also valid for one-sided limits and for limits at $\pm\infty$.

Examples:

$$\lim_{x\to 1}\frac{\ln x}{x^2-x}$$

$$\lim_{x\to 1}\frac{\frac{1}{x}}{2x-1}$$

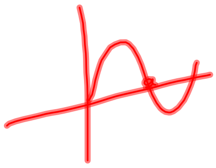
$$=\frac{1}{2\cdot 1}=1$$

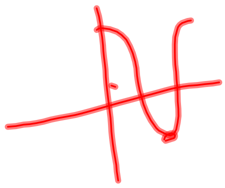
$$\lim_{x\to 0}\frac{e^{5x}-1}{3x}$$

$$\lim_{x\to 0}\frac{e^{5x}(5)}{3}$$

$$=\frac{5}{3}$$

$$\lim_{x\to 0}\frac{4x-\sin 4x}{x^3}$$





$$\lim_{x\to 0}\frac{4-\cos 4x(4)}{3x^2}$$

$$\lim_{x\to 0}\frac{\sin 4x(16)}{6x}$$

Keep Differentiating

$$\lim_{x\to 0}\frac{\cos 4x(64)}{6}$$

$$=\frac{64}{6}$$

$$=\frac{32}{3}$$

$$\lim_{x\to\pi}\frac{\sin x}{1-\cos x}$$

Not Indeterminate

$$=\frac{0}{1-(-1)}=\frac{0}{2}$$

$$=0$$

Also works for limits of quotients approaching ∞/∞

$$\lim_{x\to\infty} \frac{2x^3/x^3 - x^2/x^3 + 1/x^3}{1/x^3 - 5x^3/x^3}$$

$= -\frac{2}{5}$

$$\lim_{x\to\infty} \frac{e^x}{2x^2}$$

$\lim_{x\to\infty} \frac{e^x}{4x}$

$\lim_{x\to\infty} \frac{e^x}{4}$

$= \frac{1}{4}e^{\infty} \to \infty$

$\therefore$ Does Not exist

Practice Problems...

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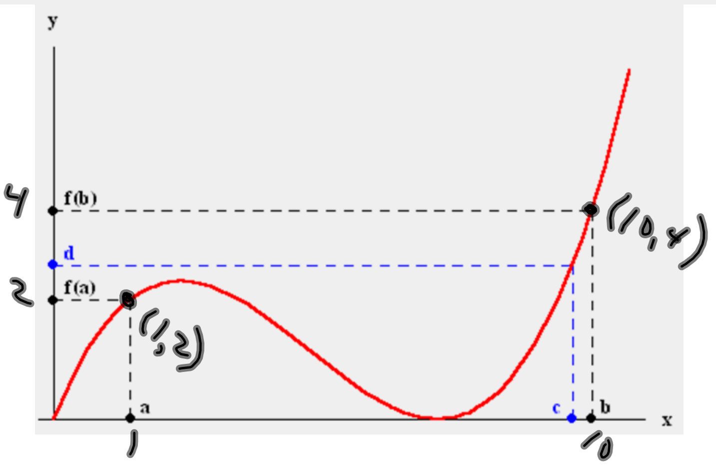
Odd numbered questions

Intermediate Value Theorem (IVT)

Intermediate Value Theorem. Let $f(x)$ be a continuous function on the interval $[a, b]$. If $d \in [f(a), f(b)]$, then there is a $c \in [a, b]$ such that $f(c) = d$.

$f(x)=3$

$f(1)=2$
$f(10)=4$



So, the Intermediate Value Theorem tells us that a function will take the value of M somewhere between a and b but it doesn't tell us where it will take the value nor does it tell us how many times it will take the value.

Example 4 Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1,2].

Solution

What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

To do this all we need to do is compute,

$$p(-1) = 8 \qquad\qquad p(2) = -19$$

$$(-1, 8) \qquad\qquad (2, -19)$$

So we have,

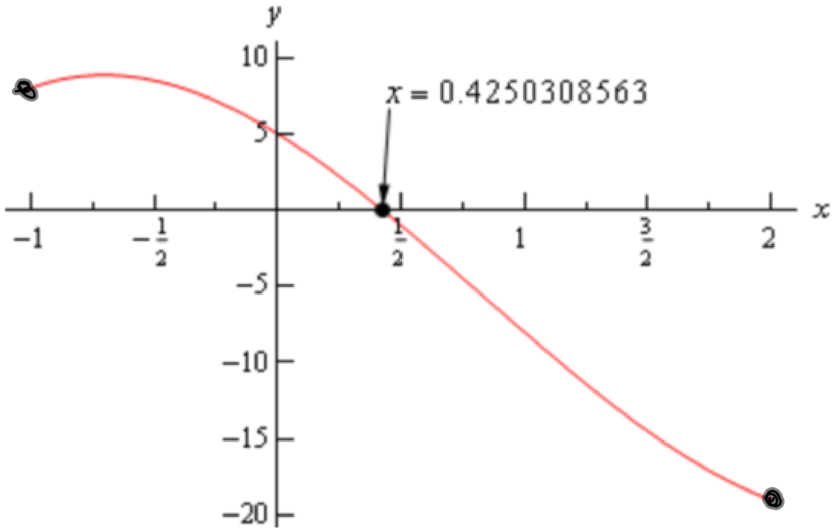
$$-19 = p(2) < 0 < p(-1) = 8$$

Therefore $M = 0$ is between $p(-1)$ and $p(2)$ and since $p(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that,

$$p(c) = 0$$

Therefore the polynomial does have a root between -1 and 2.

For the sake of completeness here is a graph showing the root that we just proved existed. Note that we used a computer program to actually find the root and that the Intermediate Value Theorem did not tell us what this value was.



Example 5 If possible, determine if $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval [0,5].

No $f(0)=2.8$ $f(5)=-15.9$ Radians

(a) Does $f(x) = 10$? [Solution]

(b) Does $f(x) = -10$? [Solution]

Yes

So, since we'll need the two function evaluations for each part let's give them here,

$$f(0) = 2.8224 \qquad f(5) = \cancel{19.7436}$$

-15.9

(a) Okay, in this case we'll define $M = 10$ and we can see that,

$$f(0) = 2.8224 < 10 < 19.7436 = f(5)$$

So, by the Intermediate Value Theorem there must be a number $0 \le c \le 5$ such that

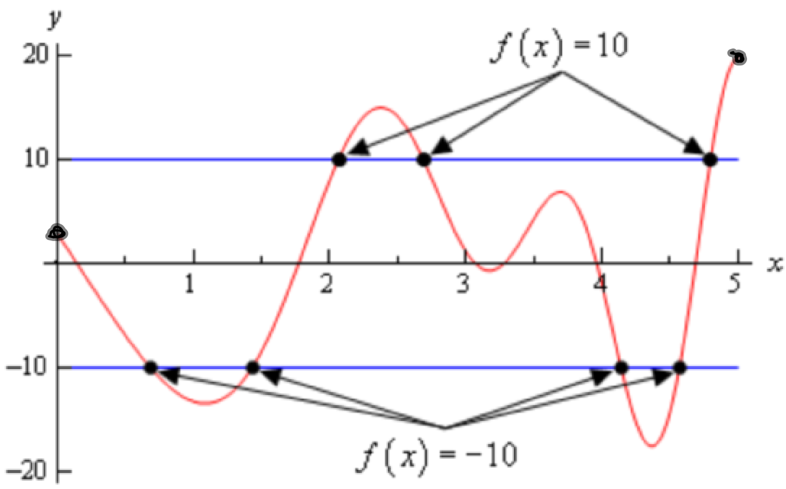
$$f(c) = 10$$

(b) In this part we'll define $M = -10$. We now have a problem. In this part M does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \neq -10$ in [0,5]?

Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \neq -10$ in [0,5], but is it also possible that $f(x) = -10$ in [0,5]. The Intermediate Value Theorem will only tell us that c's will exist. The theorem will NOT tell us that c's don't exist.

In this case it is not possible to determine if $f(x) = -10$ in [0,5] using the Intermediate Value Theorem.

For completeness sake here is the graph of $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ in the interval [0,5].



From this graph we can see that not only does $f(x) = -10$ in [0,5] it does so a total of 4 times!

Mean Value Theorem (MVT)

he **Mean Value Theorem** is one of the most important theoretical tools in Calculus. It states that if $f(x)$ is defined and continuous on ie interval $[a,b]$ and differentiable on (a,b), then there is at least one number c in the interval (a,b) (that is $a < c < b$) such that

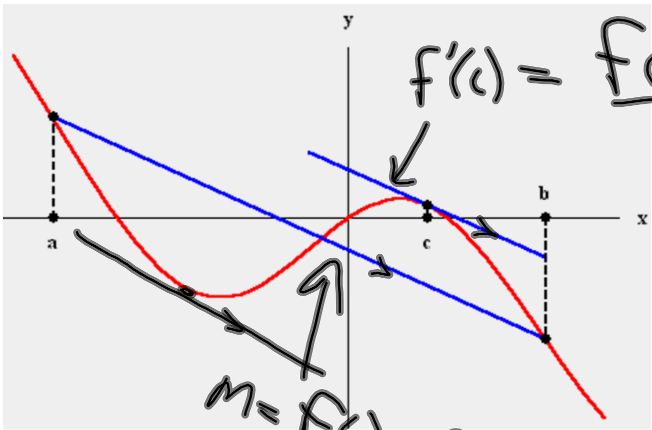
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$(a, f(a))$ & $(b, f(b))$

he special case, when $f(a) = f(b)$ is known as **Rolle's Theorem**. In this case, we have $f'(c) = 0$. In other words, there exists a point in ie interval (a,b) which has a horizontal tangent. In fact, the Mean Value Theorem can be stated also in terms of slopes. Indeed, the umber

$$\frac{f(b) - f(a)}{b - a}$$

the slope of the line passing through $(a,f(a))$ and $(b,f(b))$. So the conclusion of the Mean Value Theorem states that there exists a point $\in (a, b)$ such that the tangent line is parallel to the line passing through $(a,f(a))$ and $(b,f(b))$. (see Picture)

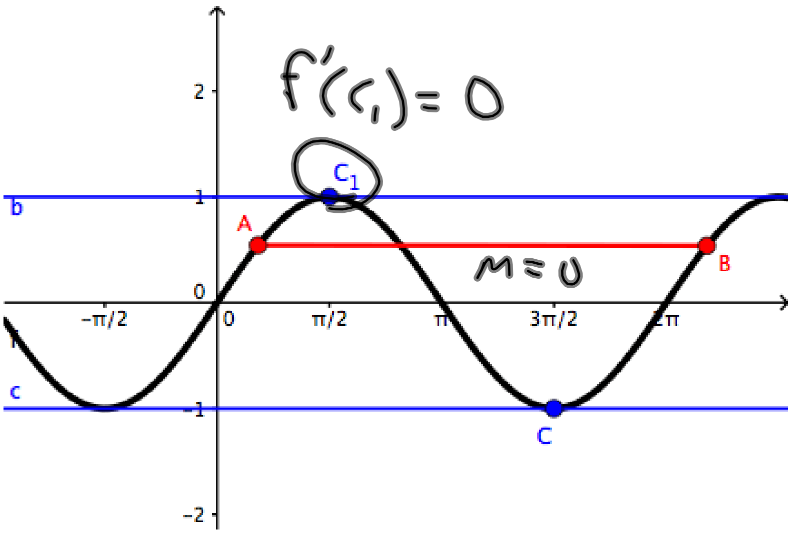


$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

$$m = \frac{f(a) - f(b)}{a - b}$$

Rolle's Theorem

Rolle's Theorem is a special case of the Mean Value Theorem. It is stating the same thing, but with the condition that f(a) = f(b). If this is the case, there is a point c in the interval [a,b] where f'(c) = 0.



Problem 1: Find a value of c such that the conclusion of the mean value theorem is satisfied for

$$f(x) = -2x^3 + 6x - 2$$

on the interval [-2 , 2]

$f(2) = -6 \quad (2,-6)$

$f(-2) = 2 \quad (-2,2)$

Slope of secant connecting endpoints

$m = \frac{-8}{4} = -2$

Slope of secant = slope of tangent

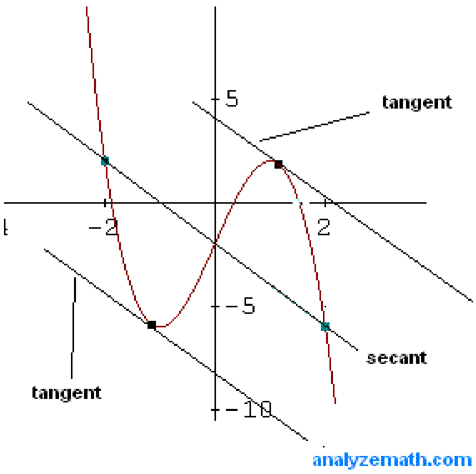
$-2 = f'(x)$

$-6x^2 + 6 = -2$

$-6x^2 = -8$

$x^2 = \frac{4}{3}$

$x = \pm \frac{2}{\sqrt{3}}$



Consider $f(x) = x^2/(x+2)$ on the interval $[-1, 1]$. Find the *exact* value of every c that satisfies the conclusion of the Mean Value Theorem.

$f(-1) = \frac{1}{1} = 1$ $\qquad$ $f(1) = \frac{1}{3}$

$(-1, 1)$ $\qquad$ $(1, \frac{1}{3})$

$$m = \frac{\frac{1}{3} - 1}{2} = \frac{-\frac{2}{3}}{2} = -\frac{2}{3} \cdot \frac{1}{2}$$
$$= -\frac{1}{3}$$

$$f(x) = \frac{x^2}{x+2}$$

$$f'(x) = \frac{2x(x+2) - x^2}{(x+2)^2}$$

$$f'(x) = \frac{2x^2 + 4x - x^2}{(x+2)^2}$$

$$f'(x) = \frac{x^2 + 4x}{(x+2)^2}$$

$$\frac{x^2 + 4x}{(x+2)^2} = \frac{-1}{3}$$

$$3x^2 + 12x = -(x+2)^2$$

$$3x^2 + 12x = -(x^2 + 4x + 4)$$

$$4x^2 + 16x + 4 = 0$$

$$x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2}$$

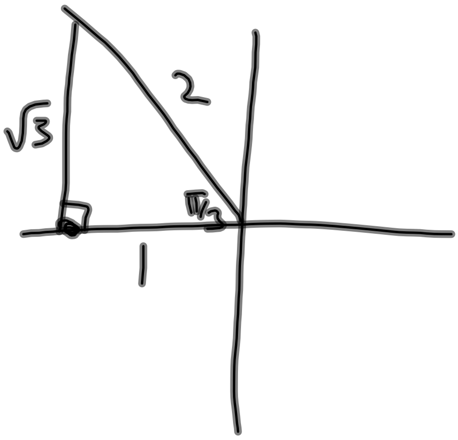
$$x = \frac{-4 \pm 2\sqrt{3}}{2}$$

$x = -2 + \sqrt{3}$ OR $x = -2 - \sqrt{3}$

Verify that $f(x) = \sin(x)$ satisfies the conditions of Rolle's Theorem on the interval $[\pi/3, 8\pi/3]$. Find the *exact* value of every number c that satisfies the conclusion of Rolle's Theorem.

$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$f\left(\frac{8\pi}{3}\right) = \sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$



$\frac{9\pi}{3} - \frac{\pi}{3}$

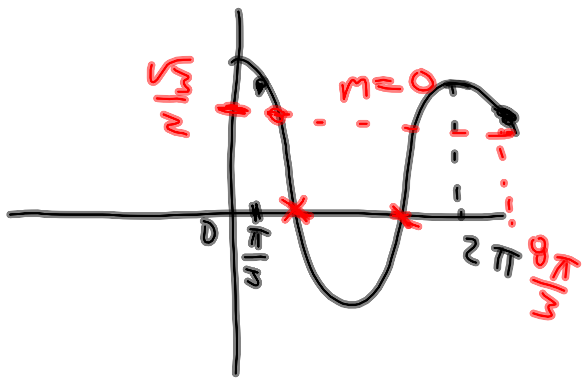
$3\pi - \frac{\pi}{3}$

slope of secant $= \dfrac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\text{whatever}} = 0$

$f'(x) = \cos x$

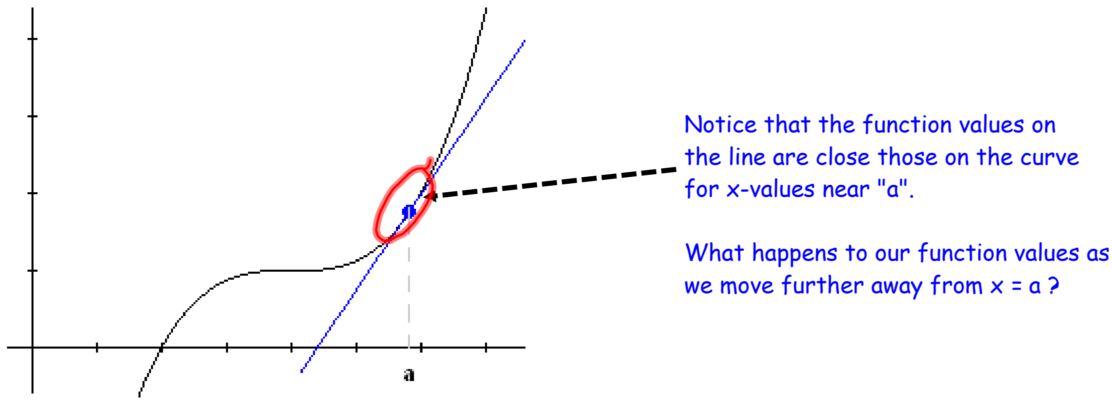
$\cos x = 0 \quad \frac{\pi}{3}, \frac{8\pi}{3}$

$x = \frac{\pi}{2}, \ \& \ \frac{3\pi}{2}$



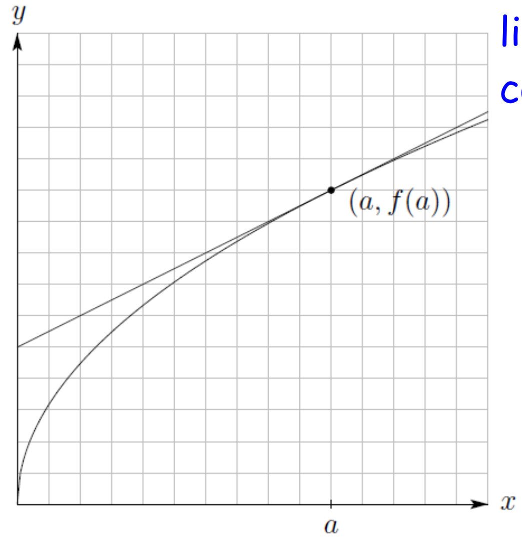
Linear Approximation

Linear approximation: Differentiation is used to approximate function values by using a linear function nearby.



A linear approximation (or tangent line approximation) is the simple idea of using the equation of the tangent line to approximate values of $f(x)$ for x near $x = a$.

Let's derive a formula that can be used to determine the linear approximation of any continuous function f(x)



point-slope formula

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a)$$

Again, the idea in linear approximation is to approximate the y values on the graph $y = f(x)$ with the y values of the tangent line $y = f(a) + f'(a)(x - a)$, so long as x is not too far away from a. That is,

$$\boxed{\text{for } x \text{ near } a,\ f(x) \approx f(a) + f'(a)(x - a)\ .}$$

Example: Determine the linear approximation for $f(x)=\sqrt[3]{x}$ at $x=8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

$f'(x)=\frac{1}{3}x^{-2/3}$

$(8, f(8))$

$f'(8)=\frac{1}{3}(8)^{-2/3}=\frac{1}{3}\left(\frac{1}{4}\right)=\frac{1}{12}$

$(8, 2)$

$y-2=\frac{1}{12}(x-8)$

$y=\frac{1}{12}x+\frac{4}{3}$