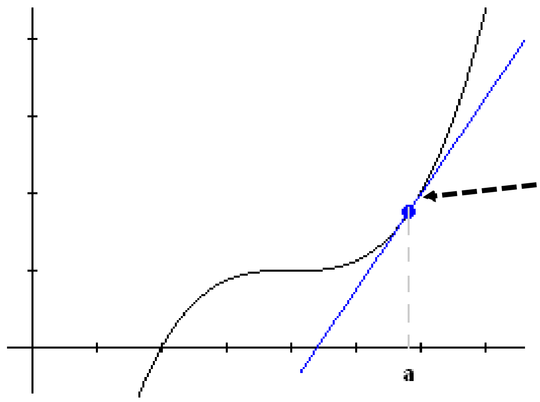
Linear Approximation

Linear approximation: Differentiation is used to approximate function values by using a linear function nearby.

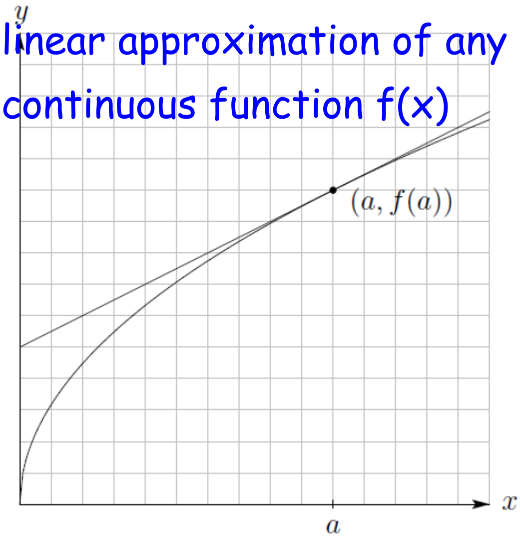


Notice that the function values on the line are close those on the curve for x-values near "a".

What happens to our function values as we move further away from $x = a$?

A linear approximation (or tangent line approximation) is the simple idea of using the equation of the tangent line to approximate values of $f(x)$ for x near $x = a$.

Let's derive a formula that can be used to determine the linear approximation of any continuous function f(x)



point-slope formula

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x - a)$$

$$y = f(a) + f'(a)(x - a)$$

Again, the idea in linear approximation is to approximate the y values on the graph $y = f(x)$ with the y values of the tangent line $y = f(a) + f'(a)(x - a)$, so long as x is not too far away from a. That is,

$$\boxed{\text{for } x \text{ near } a,\ f(x) \approx f(a) + f'(a)(x - a)\ .}$$

Example: Determine the linear approximation for $f(x)=\sqrt[3]{x}$ at $x=8$. Use the linear approximation to approximate the value of $\sqrt[3]{8.05}$ and $\sqrt[3]{25}$.

$f'(x)=\frac{1}{3}x^{-2/3}$

$f(8)=2$

$(8,2)$

$m \Rightarrow f'(8)=\frac{1}{3}(8)^{-2/3}$

$=\frac{1}{12}$

Tangent Line:

$y-2=\frac{1}{12}(x-8)$

$y=\frac{1}{12}x+\frac{4}{3}$

Approx: at 8.05

$y=\frac{1}{12}(8.05)+\frac{4}{3}$

$y=2.00417$

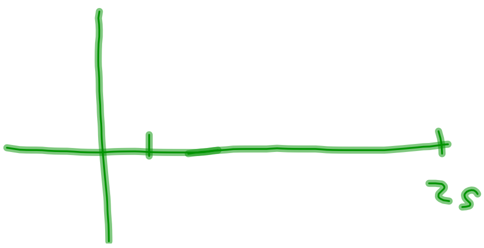
$\sqrt[3]{8.05}=2.00416$

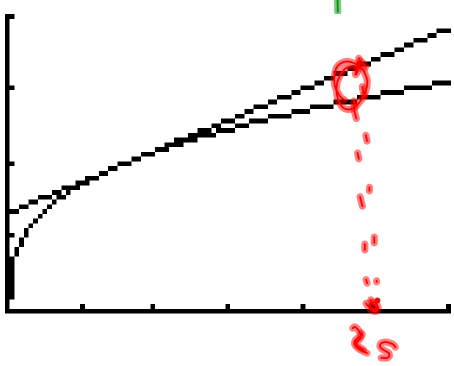
$\sqrt[3]{25}\Rightarrow x=25$

Actual: 2.924

$y=\frac{1}{12}(25)+\frac{4}{3}$

$y=3.417$





$f''(2) = -8$ $\quad$ $f''(x) < 0$ ((concave down))

concavity

$\left.\begin{array}{l} f'(2) = 3 \\ f(2) = -1 \end{array}\right\}$

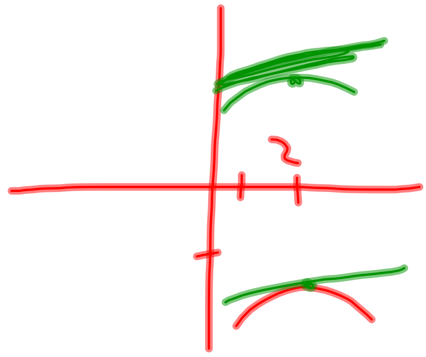
1) Approximate $f(2.1)$
2) Overestimate?
OR
Underestimate?

Tangent Line:

$(2, -1)$ & $m = 3$

$y + 1 = 3(x - 2)$

$y = 3x - 7$



estimate: $f(2.1) = 3(2.1) - 7$

$= -0.7$

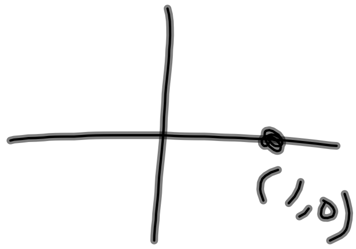
Example : Find the linear approximation of $f(x) = x\sin(\pi x^2)$ about $x = 2$. Use the approximation to estimate $f(1.99)$.

$f'(x) = \sin(\pi x^2) + x\cos(\pi x^2)(2\pi x)$

$f'(2) = \sin 4\pi + 2\cos(4\pi)(4\pi)$

$f'(2) = 0 + 2(1)(4\pi)$

$= 8\pi$



$f(2) = 2\sin(4\pi)$

$= 0$

$(2,0)$

$y - 0 = 8\pi(x-2)$

$y = 8\pi x - 16\pi$

at 1.99 . . .

$y = 8\pi(1.99) - 16\pi$

$y = -0.25132$

<u>Actual:</u>

$y = 1.99(\sin[\pi(1.99)^2])$

$y = \underline{-0.249}$

Warm Up

The approximate value of $y=\sqrt{4+\sin x}$ at $x=0.12$, obtained from the tangent to the graph at $x=0$, is

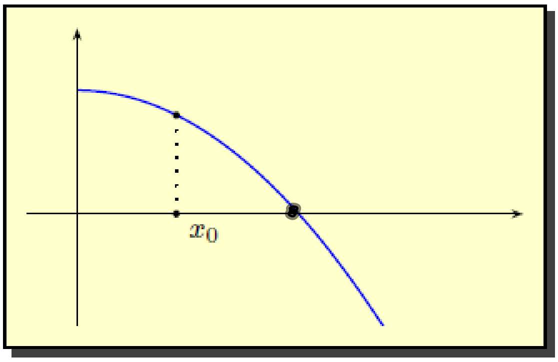
(A) 2.00 (B) 2.03 (C) 2.06 (D) 2.12 (E) 2.24

Let f be a differentiable function such that $f(3)=2$ and $f'(3)=5$. If the tangent line to the graph of f at $x=3$ is used to find an approximation to a zero of f, that approximation is

(A) 0.4 (B) 0.5 (C) 2.6 (D) 3.4 (E) 5.5

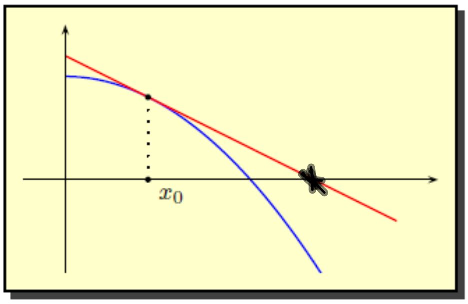
Newton's Method

Problem: Given an equation $f(x) = 0$, solve for x numerically.

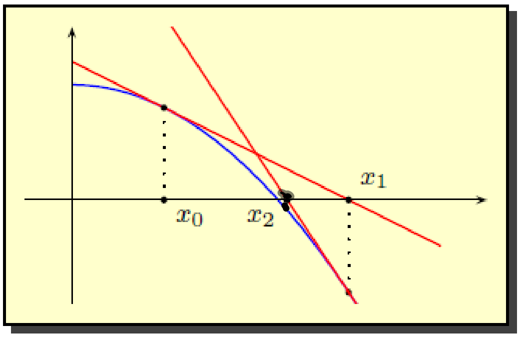


- Make an *initial guess*: x_0. Now go up to the curve.

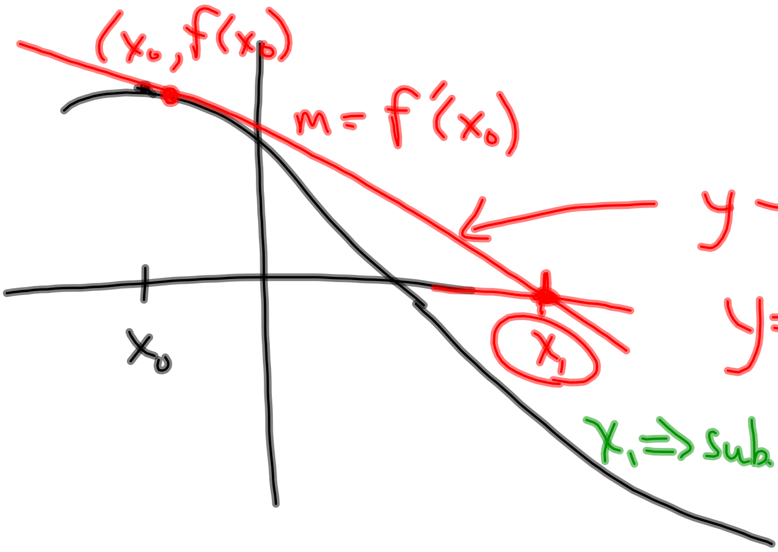
- Draw the tangent line.



- Let x_1 be in x-intercept of this tangent line.



- This intercept is given by the formula: $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$.
- Now repeat using x_1 as the initial guess.
- The intercept x_2 is given by: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.



$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$y = f'(x_0)[x - x_0] + f(x_0)$$

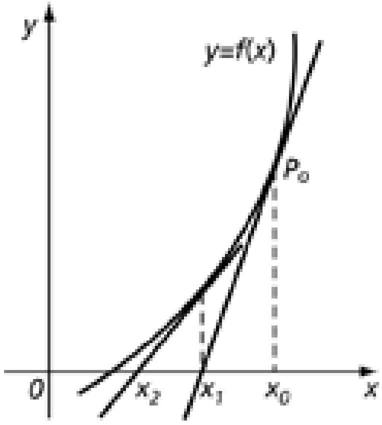
$x_1 \Rightarrow$ sub $y = 0 \ldots$

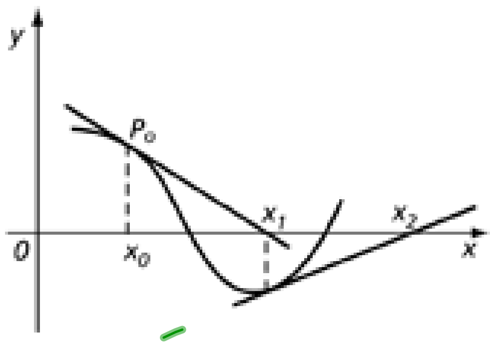
$$0 = f'(x_0)(x - x_0) + f(x_0)$$

$$\frac{-f(x_0)}{f'(x_0)} + x_0 = x$$

Newton's Method:

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$


y
y=f(x)
P₀
0
x₂
x₁
x₀
x


y
P₀
x₁
x₂
0
x₀
x

Newton Iteration formula: $$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

The initial guess x_0, and the Newton Iteration formula, together form an *algorithm* or a procedure of estimating the value of the root to the equation $f(x) = 0$.

Find the positive root of the equation $x^2 = 2$.

$f(x) = x^2 - 2$

$x^2 - 2 = 0$

guess: $x = 1$

$f'(x) = 2x$

$x_0 = 1$

$x_1 = 1 - \left(\frac{-1}{2}\right)$

$x_1 = 1.5$

$x_2 = 1.5 - \left(\frac{0.25}{3}\right)$

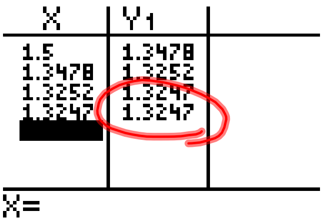
$x_2 = 1.416$

$x_3 = 1.416 - \_\_\_\_$

Newton's Method		
$f(x) = x^2 - 2, \quad x_0 = 1.5$		
n	x_n	$f(x_n)$
0	1.50000000	0.25000000
1	1.41666667	0.00694445
2	1.41421568	0.00000600
3	1.41421356	0.00000000
4	1.41421356	0.00000000

Example 3.2. Find a solution to the equation $x^3 = x + 1$ that is near $x_0 = 1.5$.

$$\underbrace{x^3 - x - 1}_{f(x)} = 0$$



x = 1.3247

Newton's Method		
$f(x) = x^3 - x - 1, \quad x_0 = 1.5$		
n	x_n	$f(x_n)$
0	1.50000000	0.87500000
1	1.34782608	0.10058217
2	1.32520039	0.00205836
3	1.32471817	0.00000092
4	1.32471795	0.00000000
5	1.32471795	0.00000000

Practice Problems...

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