

Sigma Notation

$$\sum_{k=1}^n f(x_k)$$

Example:

Evaluate $\sum_{k=1}^4 (3k - 2)$

What about the following?

$$\sum_{k=1}^7 (5w) \quad \sum_{k=1}^5 (-3)$$

Summation properties...

$$\sum_{i=1}^n C = Cn$$

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n Ca_i = C \sum_{i=1}^n a_i$$

Evaluate the following...

- (i) without summation properties
- (ii) using summation properties

$$\sum_{i=1}^3 (2i^2 - 5i)$$

Summation Rules

$$\sum_{k=1}^n k = 1 + 2 + 3 + 4 + \dots + (n-1) + n$$
$$\sum_{k=1}^n k^2 = 1 + 4 + 9 + 16 + \dots + (n-1)^2 + n^2$$
$$\sum_{k=1}^n k^3 = 1 + 8 + 27 + 64 + \dots + (n-1)^3 + n^3$$

Notice that these
are INFINITE

$$\sum_{k=1}^n k = \frac{n^2}{2} + \frac{n}{2}$$
$$\sum_{k=1}^n k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$
$$\sum_{k=1}^n k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

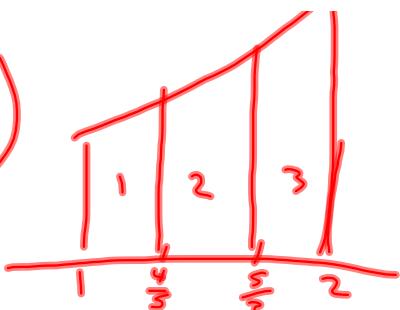
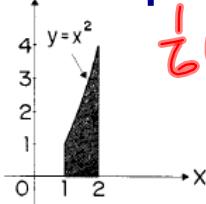
Notice that these
are FINITE

$$\frac{8}{9} + \frac{5}{1}$$

Warm-Up

$$A = \frac{1}{2} \left(\frac{1}{3} \right) \left[\left(1 + \frac{16}{9} \right) + \left(\frac{16}{4} + \frac{25}{9} \right) + \left(\frac{25}{9} + 4 \right) \right]$$

$$\frac{1}{2} \left(\frac{127}{9} \right)$$



Calculate the approximate area of the shaded region in the figure by the trapezoidal rule, using divisions at $x = \frac{4}{3}$ and $x = \frac{5}{3}$.

(A) $\frac{50}{27}$

(B) $\frac{251}{108}$

(C) $\frac{7}{3}$

(D) $\frac{127}{54}$

(E) $\frac{77}{27}$

Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(1) = 0$, then $F(9) =$

(A) 0.048

(B) 0.144

(C) 5.827

(D) 23.308

(E) 1,640.250

$$f(x) = (\ln x)^3 \left(\frac{1}{x} \right)$$

$$F(x) = \frac{1}{4} (\ln x)^4 + C$$

$$0 = \frac{1}{4} (\ln 1)^4 + C$$

$$0 = C$$

$$F(9) = \frac{1}{4} (\ln 9)^4$$

$$= \underline{\underline{5.827}}$$

Let's revisit the example used yesterday...

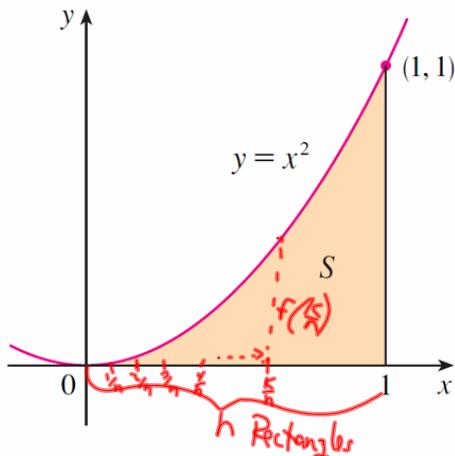


FIGURE 3

We want to find the area below this curve using "n" rectangles.

$$\text{What will be the width of each rectangle? } \Delta x = \frac{1}{n}$$

$$\text{How will we determine the height of each rectangle? } x_k = \frac{k}{n}$$

$$\text{height} = f\left(\frac{k}{n}\right)$$

Write out an expression for the area of these "n" rectangles?

$$A = \sum_{k=1}^n \left(\frac{1}{n}\right) f\left(\frac{k}{n}\right)$$

$$A = \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^2$$

$$A = \frac{1}{n} \sum_{k=1}^n \frac{k^2}{n^2}$$

$$A = \frac{1}{n} \sum_{k=1}^n \left(\frac{1}{n}\right) k^2$$

$$A = \frac{1}{n^3} \sum_{k=1}^n k^2 \quad \text{Sub. Summation Rule}$$

$$\begin{aligned} \sum_{k=1}^n k &= \frac{n^2}{2} + \frac{n}{2} \\ \sum_{k=1}^n k^2 &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ \sum_{k=1}^n k^3 &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \end{aligned}$$

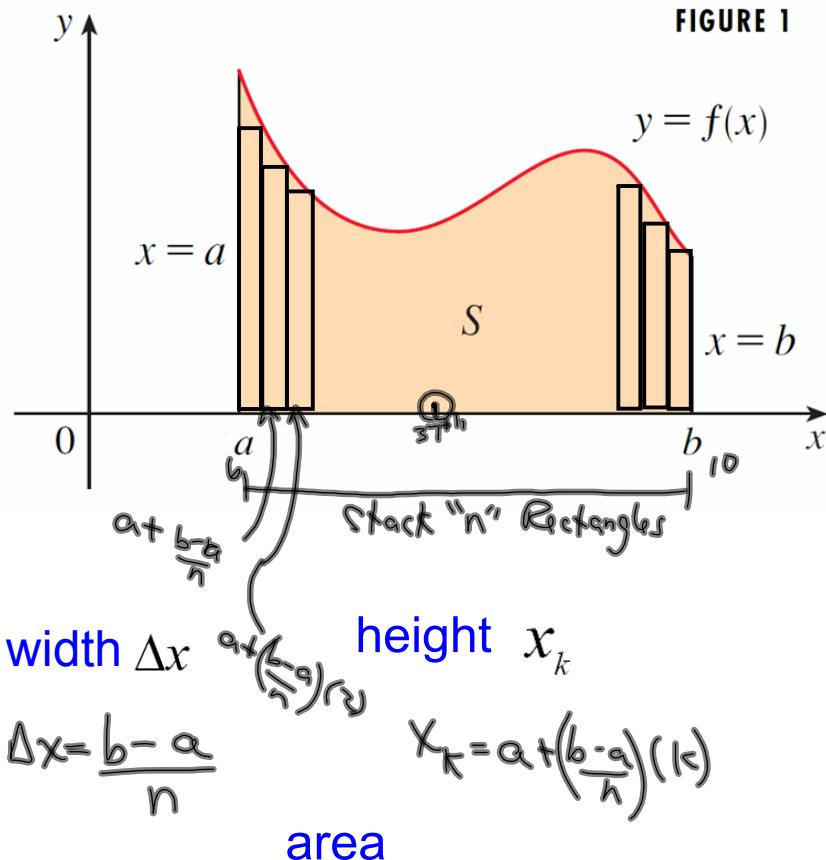
$$A = \frac{1}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \leftarrow \dots \text{What is } n??$$

Actual Area ... let $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} \right)$$

$$= \frac{1}{3} u^2$$

Develop a general formula for the area below a curved surface using "n" rectangles.



$$A = \sum_{k=1}^n \Delta x f(x_k)$$

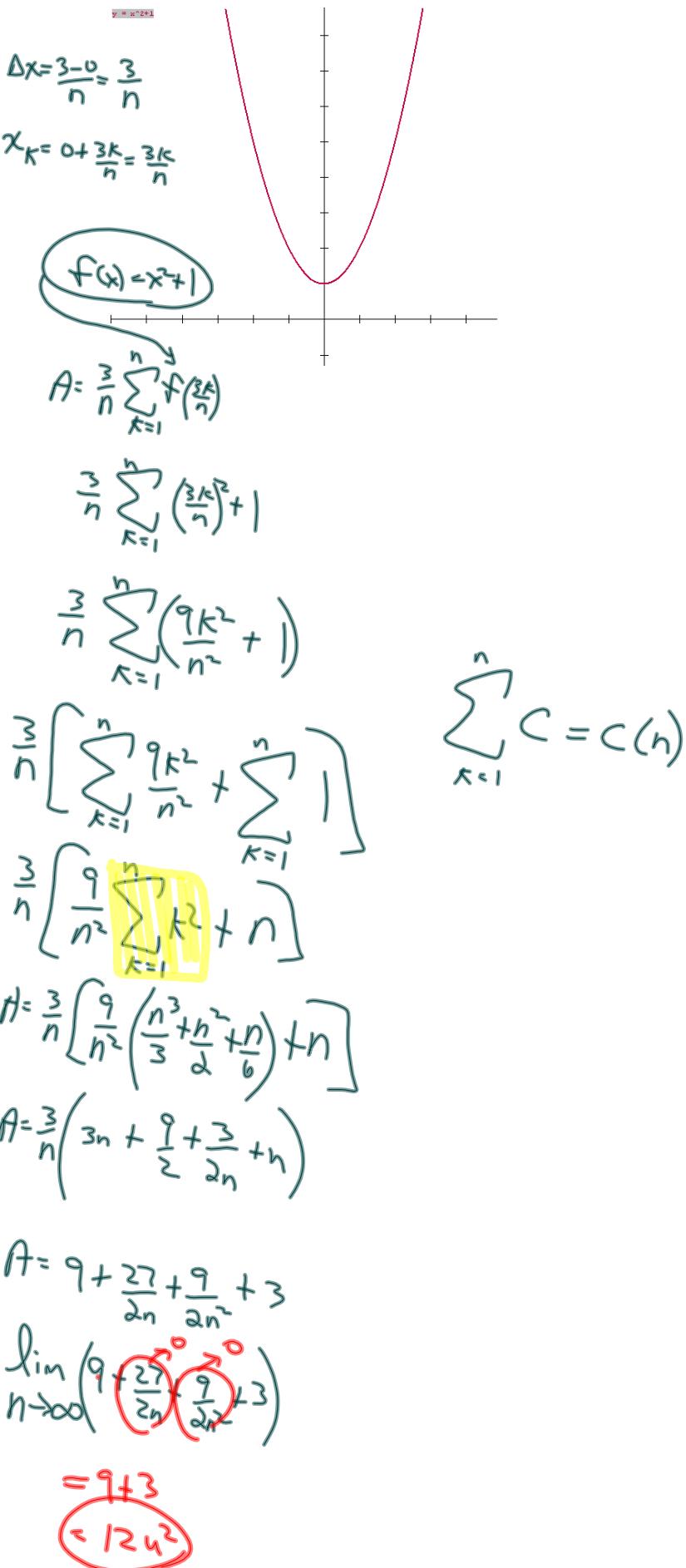
$\approx \Delta x \sum_{k=1}^n f(x_k)$

$\left\{ \begin{array}{l} \Delta x = \frac{b-a}{n} \\ x_k = a + (\Delta x)k \end{array} \right.$

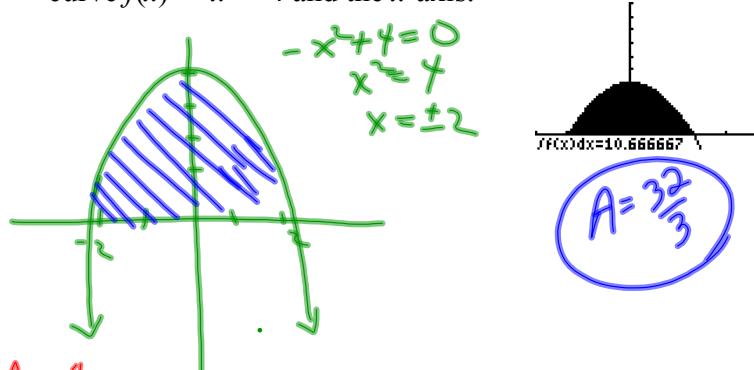
$$A = \Delta x \sum_{k=1}^n f(x_k)$$

$$\Delta x = \frac{b-a}{n} \quad x_k = a + (\Delta x)k$$

Use a Riemann Summation to determine the area below the curve $y = x^2 + 1$, between $x = 0$ and $x = 3$.



Use a Riemann sum to determine the area bound by the curve $f(x) = -x^2 + 4$ and the x -axis.



$$\Delta x = \frac{4}{n} \quad x_k = -2 + \frac{4k}{n} \quad f(x) = -x^2 + 4$$

$$A = \frac{4}{n} \sum_{k=1}^n f\left(-2 + \frac{4k}{n}\right)$$

$$A = \frac{4}{n} \sum_{k=1}^n -\left(-2 + \frac{4k}{n}\right)^2 + 4$$

$$A = \frac{4}{n} \sum_{k=1}^n \left[-\left(4 - \frac{16k}{n} + \frac{16k^2}{n^2}\right) + 4 \right]$$

$$A = \frac{4}{n} \sum_{k=1}^n \left(\frac{16k}{n} - \frac{16k^2}{n^2} \right)$$

$$A = \frac{4}{n} \left[\frac{16}{n} \sum_{k=1}^n k - \frac{16}{n^2} \sum_{k=1}^n k^2 \right]$$

$$A = \frac{4}{n} \left[\frac{16}{n} \left(\frac{n^2}{2} + \frac{n}{2} \right) - \frac{16}{n^2} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \right]$$

$$A = \frac{4}{n} \left(8n + 8 - \frac{16n}{3} - \frac{16}{6} - \frac{16}{6n} \right)$$

$$A = \left[\frac{32}{3} - \frac{64}{6n} \right] - \frac{64}{6n^2}$$

$$\lim_{n \rightarrow \infty} \left(\frac{32}{3} - \frac{64}{6n^2} \right)$$

$$A = \frac{32}{3}$$

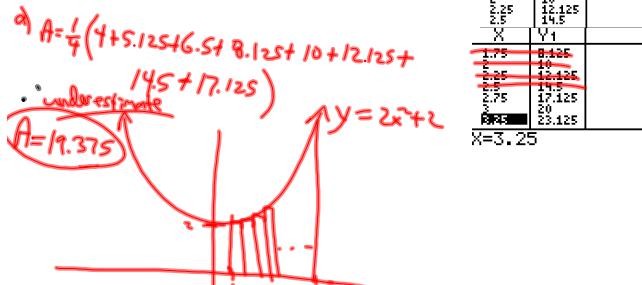
Example:

Suppose R is the area in the first quadrant below the curve

$$y = 2x^2 + 2 \text{ between } x = 1 \text{ and } x = 3.$$

- (a) Approximate the area of "R" using 8 rectangles with height determined by the left-hand endpoint

- (b) Determine the exact area of "R" using a Riemann Summation.



$$\text{b)} A = \Delta x \sum_{k=1}^n f(x_k)$$

$$\Delta x = \frac{3-1}{n} = \frac{2}{n} \quad x_k = 1 + \frac{2k}{n}$$

$$\left\{ \begin{array}{l} \Delta x = \frac{b-a}{n} \\ x_k = a + (\Delta x)k \end{array} \right.$$

$$A = \frac{2}{n} \sum_{k=1}^n f\left(1 + \frac{2k}{n}\right)$$

$$A = \frac{2}{n} \sum_{k=1}^n 2\left(1 + \frac{2k}{n}\right)^2 + 2$$

$$A = \frac{2}{n} \sum_{k=1}^n 2\left(1 + \frac{4k}{n} + \frac{4k^2}{n^2}\right) + 2$$

$$A = \frac{2}{n} \sum_{k=1}^n \left(\frac{8k^2}{n^2} + \frac{8k}{n} + 4 \right)$$

$$A = \frac{2}{n} \left[\frac{8}{n^2} \sum_{k=1}^n k^2 + \frac{8}{n} \sum_{k=1}^n k + \sum_{k=1}^n 4 \right]$$

$$A = \frac{2}{n} \left[\frac{8}{n^2} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) + \frac{8}{n} \left(\frac{n^2}{2} + \frac{n}{2} \right) + 4n \right]$$

$$A = \frac{2}{n} \left(\frac{8n}{3} + 4 + \frac{4}{3n} + 4n + 4 + 4n \right)$$

$$A = \frac{2}{n} \left(\frac{8n}{3} + \frac{4}{3n} + 8n + 8 \right)$$

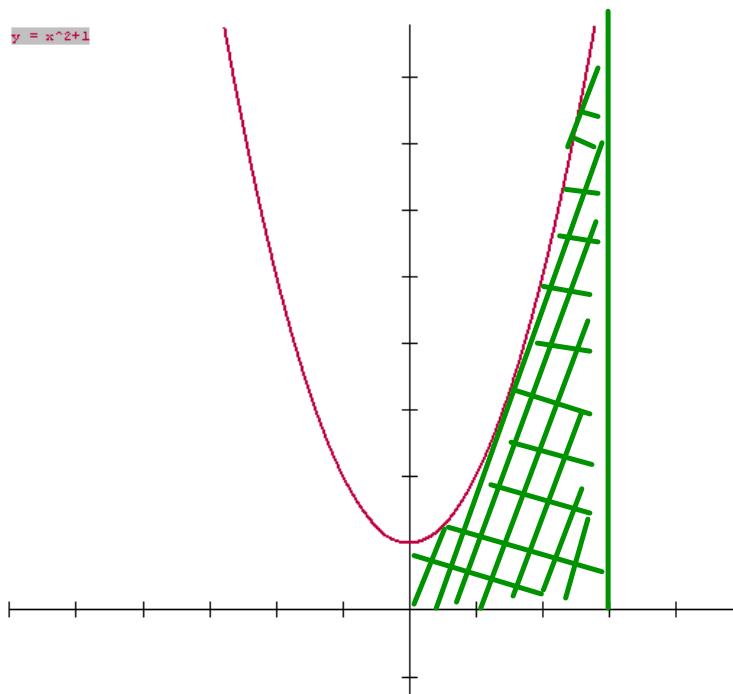
$$A = \frac{16}{3} + \frac{8}{3n^2} + 16 + \frac{16}{n}$$

$$\lim_{n \rightarrow \infty} \left(\frac{16}{3} + \frac{8}{3n^2} + 16 + \frac{16}{n} \right)$$

$$A = \frac{16}{3} + 16 = \frac{64}{3} = 21\frac{1}{3} \text{ u}^2$$

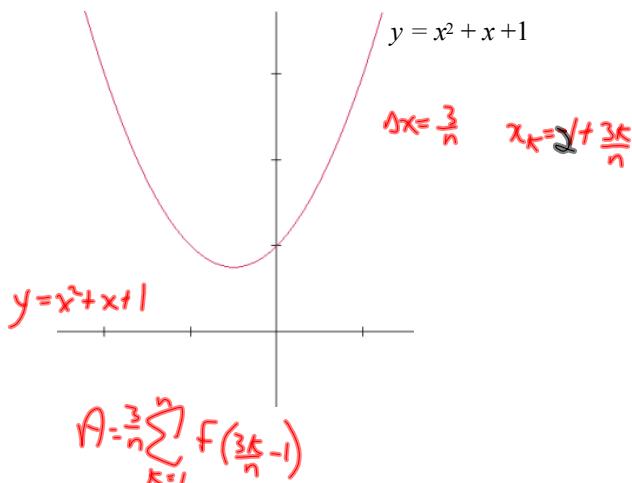
Homework Question:

Use a Riemann Summation to determine the area below the curve $y = x^2 + 1$, between $x = 0$ and $x = 3$.



Example:

Use a Riemann Summation to determine the area below the curve $y = x^2 + x + 1$ and above the x-axis, between $x = -2$ and $x = 1$.



$$A = \frac{3}{n} \sum_{k=1}^n \left(\left(\frac{3k}{n} - 1 \right)^2 + \left(\frac{3k}{n} - 1 \right) + 1 \right)$$

$$A = \frac{3}{n} \sum_{k=1}^n \left(\frac{9k^2}{n^2} - \frac{6k}{n} + 1 + \frac{3k}{n} - 1 + 1 \right)$$

$$A = \frac{3}{n} \sum_{k=1}^n \left(\frac{9k^2}{n^2} - \frac{3k}{n} + 1 \right) \quad \text{Lucas' Fault!!}$$

$$A = \frac{3}{n} \left[\sum_{k=1}^n k^2 - \frac{3}{n} \sum_{k=1}^n k + \sum_{k=1}^n 1 \right]$$

$$A = \frac{3}{n} \left[\frac{1}{n^2} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) - \frac{3}{n} \left(\frac{n^2}{2} + \frac{n}{2} \right) + n \right]$$

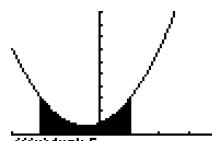
$$A = \frac{3}{n} \left(3n + \frac{9}{2} + \frac{3}{2} - \frac{3n}{2} - \frac{3}{2} + n \right)$$

$$A = \left(9 + \frac{27}{2n} + \frac{9}{2n^2} - \frac{9}{2} - \frac{9}{2n} + 3 \right)$$

$$A = \frac{9}{2n^2} + \frac{18}{2n} + 3$$

$$\lim_{n \rightarrow \infty} \left(\frac{9}{2n^2} + \frac{18}{2n} + 3 \right)$$

$$= \frac{27}{2} + 3$$



Practice Exercises:

Determine each of the following areas using a Riemann Sum with an infinite number of rectangles:

(1) Area below $f(x) = 2x^2 - 1$ between $x = 1$ and $x = 3$.

(2) Area below $f(x) = x^2 - 4x + 7$ between $x = -1$ and $x = 2$.

(3) Area below $f(x) = x^3 + 2x^2 + x$ between $x = 0$ and $x = 1$.

Advanced Placement Exam:

AP Central

{ Calculus AB \rightarrow Wed. May 9
Fee: \$82 (us)

