Applying Integrals to Rate of Change

Example 1:

If V(t) is the volume of water in a reservoir at any time *t*, then its derivative is the rate at which water flows into the reservoir at time *t*.

What would the following integral represent?

$$\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$$

This integral would represent the change in the amount of water in the reservoir between t_1 and t_2.

Example 2:

If an object moves along a straight line with a position function s(*t*), then its velocity is v(*t*) = s'(t).

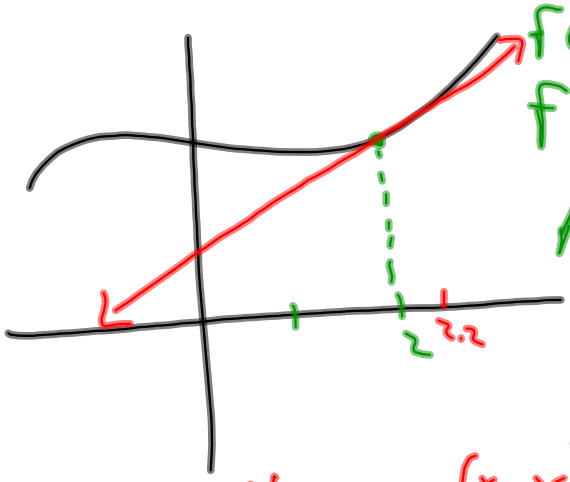
What would the following integral represent?

$$\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)$$

This would represent the net change in ~~displacement~~ position, of the particle during the time period from t_1 to t_2.

displacement

Linear Approximation



$f(2) = 3$

$f'(2) = \frac{1}{2}$

Approx. $f(2.2)$ using tangent line approximation at $x=2$.

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{1}{2}(x-2)$

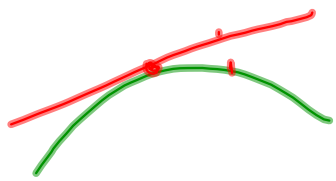
$y = \frac{1}{2}x - 1 + 3$

$y = \frac{1}{2}x + 2$ ← At $x = 2.2$

$y = \frac{1}{2}(2.2) + 2$

$y = 3.1$ ← underestimate

$f''(x)$ @ 2 is < 0



- If we want to calculate the <u>distance traveled</u> during the time interval, we have to consider the intervals both when
  - $v(t) \geq 0$ (the particle moves to the right) and
  - $v(t) \leq 0$ (the particle moves to the left).
- In both cases the distance is computed by integrating $|v(t)|$, the speed. Therefore

$$\int_{t_1}^{t_2} |v(t)|\,dt = \text{ total distance traveled}$$

Example:

A particle moves along a line so that its velocity at any time t is given by $v(t) = t^2 - t - 6 \ {}^{m}\!/_{s}$.

(a) Determine the displacement of the particle over the interval $1 \leq t \leq 4$ seconds.

(b) Determine the distance traveled over the same time interval.

Look at a diagram mapping the particle's path as well

(a) $\int_1^4 (t^2 - t - 6)\,dt$

$= \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4$

$= \left(\frac{64}{3} - 8 - 24\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$

$= \frac{64}{3} - \frac{1}{3} - 26 + \frac{1}{2}$

$= \frac{63}{3} + \frac{1}{2} - 26$

$= \frac{-5}{1} + \frac{1}{2}$

$= \frac{-9}{2}$ units

b) $t^2 - t - 6 = 0$

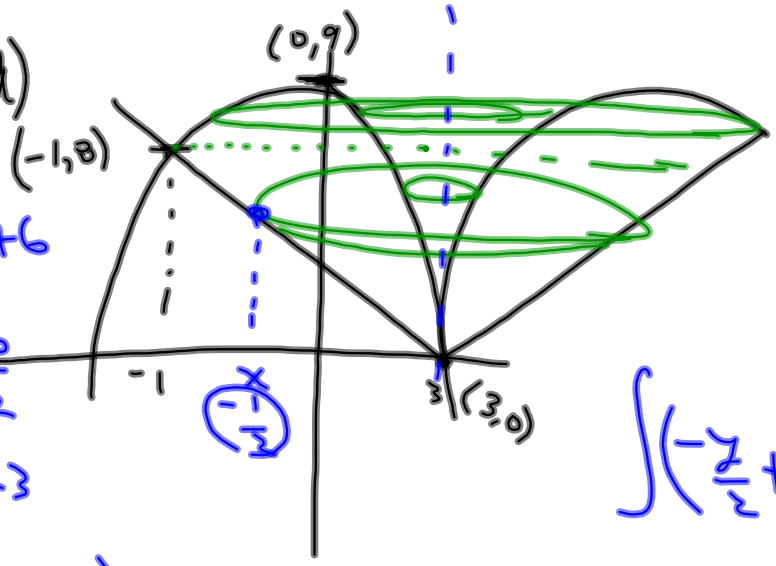
$(t-3)(t+2) = 0$

$t = 3, \cancel{-2}$

$\left| \int_1^3 (t^2 - t - 6)\,dt \right| + \left| \int_3^4 (t^2 - t - 6)\,dt \right|$

t	$s(t)$
1	
3	
4	

#3) d)



$y=-2x+6$

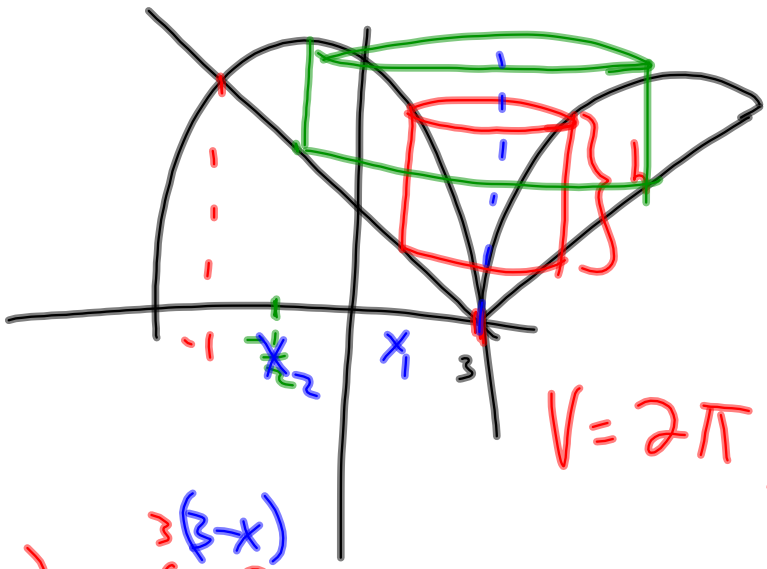
$x=\frac{y-6}{-2}$

$x=-\frac{y}{2}+3$

$x=-\frac{1}{2}$

$\int\left(-\frac{y}{2}+3\right)^2$

Do Not Recommend Washer Method !!



$$V = 2\pi \int r\,h\,(dy/dx)$$

$$2\pi \int_{-1}^{3} \overset{(3-x)}{x}\left[(9-x^2)-(-2x+6)\right]dx$$

4(b)

$$-\frac{3x^2}{1+25x^6} + \frac{5x}{x^2-1}$$

$$d(\tan^{-1} u) = \frac{du}{1+u^2}$$

$$-\frac{3}{15}\frac{(15)x^2}{1+(5x^3)^2}$$

$$\frac{5}{2}\frac{(2x)}{(x^2-1)}$$

$$d(\ln u) = \frac{du}{u}$$

$$= -\frac{1}{5}\tan^{-1} 5x^3 + \frac{5}{2}\ln(x^2-1) + C$$

Fundamental Theorem of Calculus

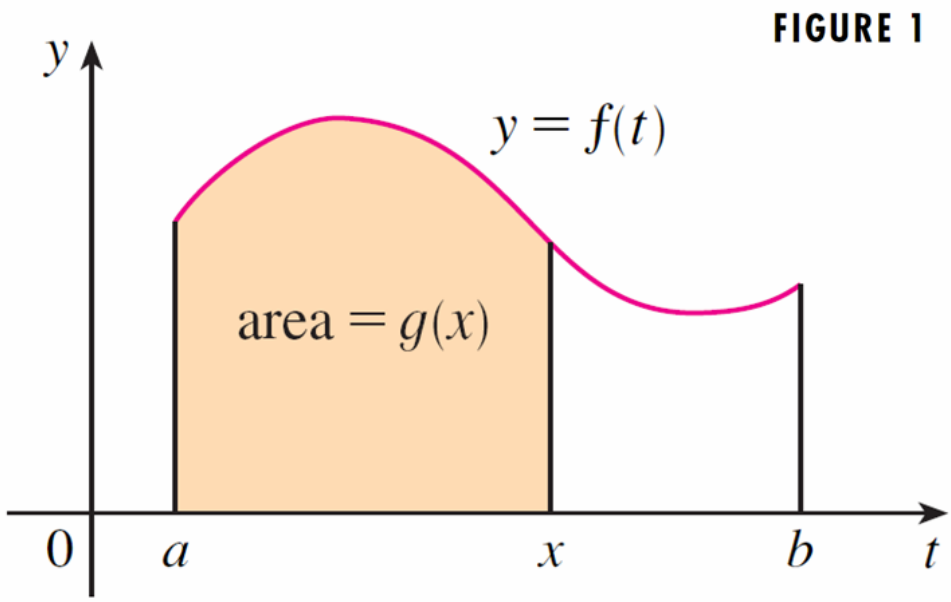
- The first part of the FTC deals with functions of the form

$$g(x) = \int_a^x f(t)\,dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b.

- If f happens to be a <u>positive</u> function, then what would $g(x)$ represent??

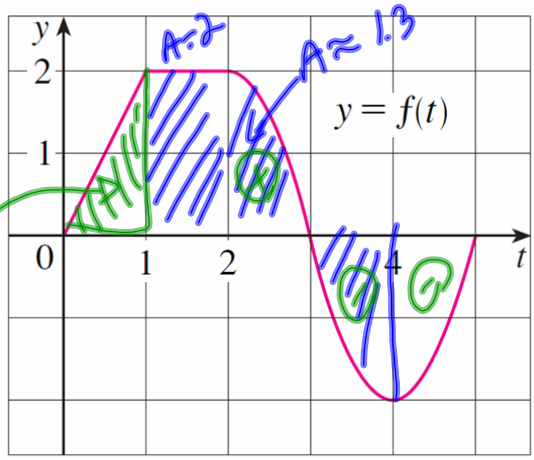
...the area under the graph of f from a to x, where x can vary from a to b.



If f is the function shown below and $g(x)=\int_a^x f(t)\,dt$, find the values of $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$ and $g(5)$.

$g(0)=0$

$g(1)=1$



$g(2)=3$

$g(3)\doteq 4.3$

$g(4)=3$

$g(5)=3-1.3$

$\doteq 1.7$

FIGURE 2

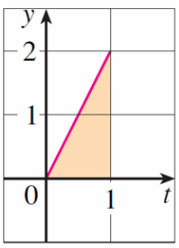
Let's look at g(0) and g(1)...

$$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = 1 + (1 \cdot 2) = 3$$

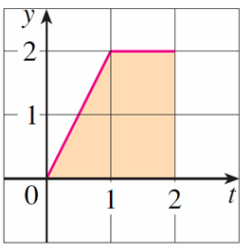
$$g(3) = g(2) + \int_2^3 f(t)\,dt \approx 3 + 1.3 = 4.3$$

$$g(4) = g(3) + \int_3^4 f(t)\,dt \approx 4.3 + (-1.3) = 3.0$$

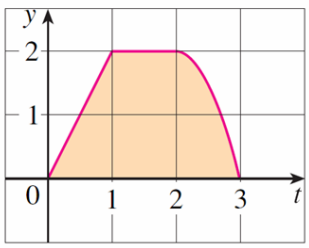
$$g(5) = g(4) + \int_4^5 f(t)\,dt \approx 3 + (-1.3) = 1.7$$



$g(1) = 1$



$g(2) = 3$



$g(3) \approx 4.3$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is an antiderivative of f, that is, $g'(x) = f(x)$ for $a < x < b$.

In Leibniz notation...

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$a \in R$

This is saying that integration and differentiation are inverses of one another.

Evaluate the following:

$$\frac{d}{dx}\int_1^x t^3\,dt$$

According to above, should equal?? x^3

Let's check using our traditional methods...

- Integrate and then differentiate

$$\int_1^x t^3\,dt$$

$$= \frac{t^4}{4}\Big|_1^x \Rightarrow \left(\frac{x^4}{4} - \frac{1}{4}\right)$$

$$\underline{\underline{der.}} \Rightarrow \frac{4x^3}{4} = \underline{x^3}$$

More Examples:

$$\frac{d}{dx}\int_{-\pi}^x \cos t\,dt = \cos x$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

$$f(x) = \frac{1}{5}\underbrace{\sec 5x \tan 5x}^{(5)} - \frac{1}{6}\underbrace{4(3x^2-5)^{10}}_{u^n \cdot du} + \frac{3}{50}\frac{\cancel{3x}\ 50x}{25x^2+1} \quad \frac{du}{u}$$

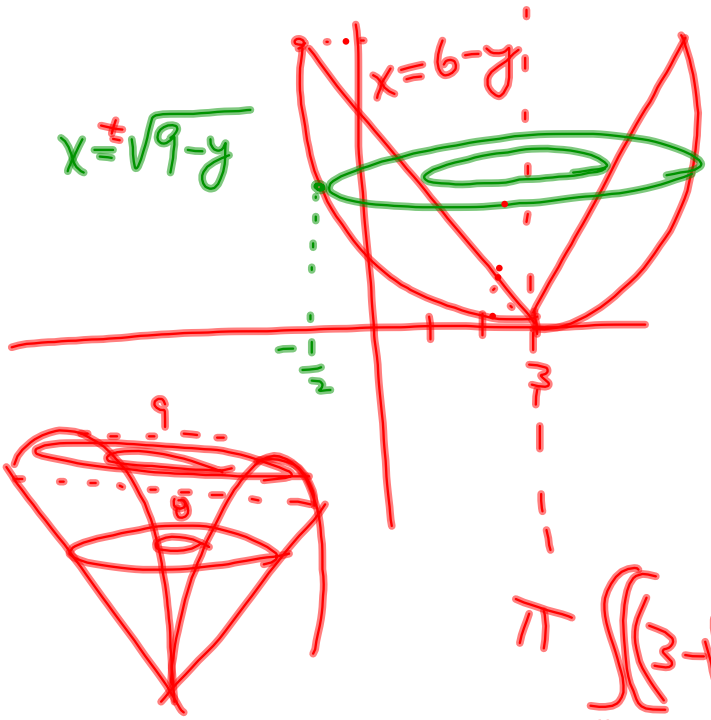
$$= \frac{1}{5}\sec 5x - \frac{1}{66}(3x^2-5)^{11} + \frac{3}{50}\ln(25x^2+1)$$

$$+ \frac{7x^3}{\sqrt{1-9x^4}} \quad -\frac{2}{3}\frac{\cancel{2x^2}\ 3x^2}{\sqrt{1-x^6}}$$

$$\frac{7}{-36}(1-9x^4)^{-1/2}(-36x^3) \qquad \sqrt{1-(x^3)^2}$$

$$u^n \cdot du$$

$$-\frac{7}{18}(1-9x^4)^{1/2} + \frac{2}{3}\sin^{-1}x^3$$

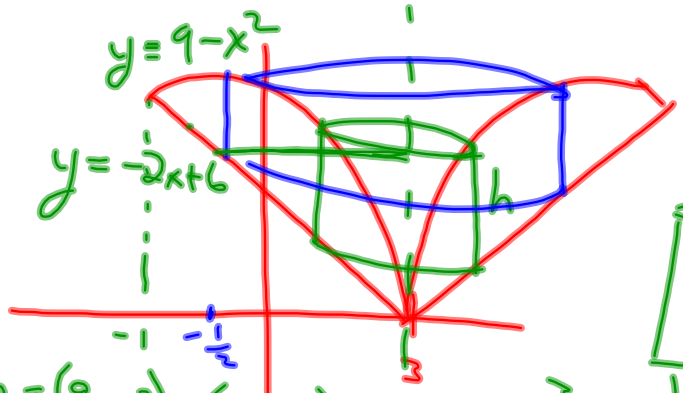


$$R = 3 - \sqrt{9-y}$$

$$r = 3 - (6-y)$$

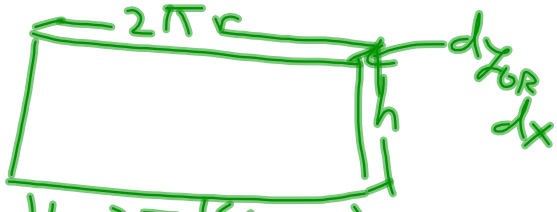
$$r = -3 + y$$

$$\pi \int_0 \left[\left(3 - \sqrt{9-y}\right)^2 - (y-3)^2 \right] dy$$



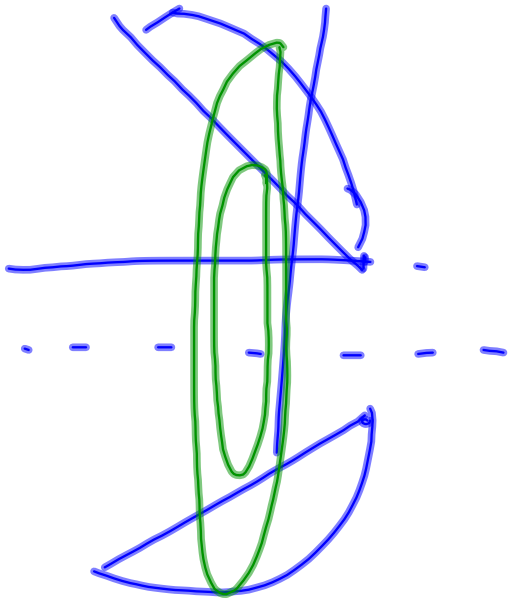
$h = (9 - x^2) - (-2x + 6)$

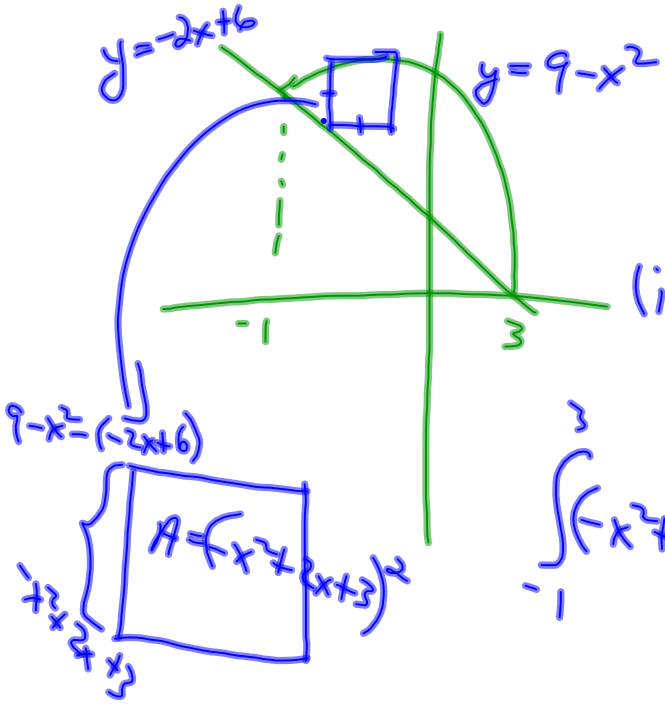
$h = -x^2 + 2x + 3$



$V = 2\pi r h \, (dy \text{ or } dx)$

$$2\pi \int_{-1}^{3} (3 - x)(-x^2 + 2x + 3)\, dx$$



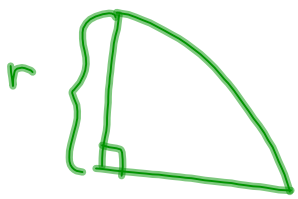


cross-sections $\perp$ to x-axis

(i) Squares

$$\int_{-1}^{3}(-x^2+2x+3)^2\,dx$$

(ii) Quarter-circles



$$A=\frac{1}{4}\pi r^2$$

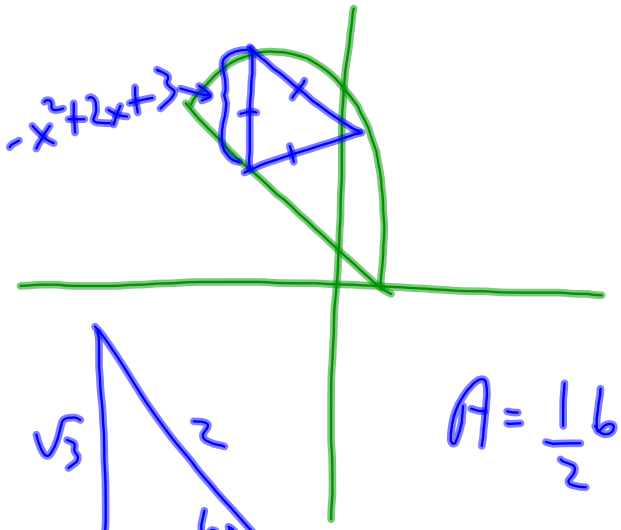
$$A=\frac{1}{4}\pi(-x^2+2x+3)^2$$

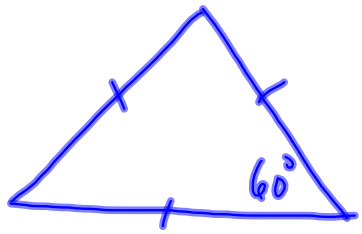
$$V=\frac{\pi}{4}\int_{-1}^{3}(-x^2+2x+3)^2\,dx$$

$$A=\frac{1}{2}\pi r^2$$

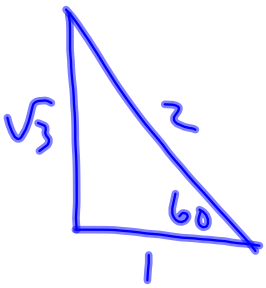
$$A=\frac{1}{2}\pi\left(\frac{1}{2}\right)(-x^2+2x+3)$$

$$V=\frac{\pi}{4}\int_{-1}^{3}(-x^2+2x+3)^2\,dx$$





$-x^2+2x+3$



$A = \frac{1}{2}bh$ OR $A = \frac{1}{2}ab\sin\theta$

$$\frac{1}{2}(-x^2+2x+3)^2 \sin 60°$$

$$\frac{\sqrt{3}}{4}\int_{-1}^{3}(-x^2+2x+3)^2\,dx$$

upper bound is not x...now what?

$$\frac{d}{dx}\int_{1}^{x^2}\cos t\,dt$$

Let $u = x^2$ and apply the chain rule when finding $\frac{dy}{dx}$ ...

$$\frac{dy}{dx} = \frac{dy}{du}\bullet\frac{du}{dx}$$

$$\frac{dy}{dx} = \left(\frac{d}{du}\int_{1}^{u}\cos t\,dt\right)\bullet\frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u\bullet\frac{du}{dx}$$

$$\frac{dy}{dx} = \cos x^2\bullet 2x$$

derivative of upper bound

Now let's try and do these a little quicker...

$t = x^2$

$$\frac{d}{dx}\int_{0}^{x^2} e^{t^2}\,dt = 2xe^{x^4}$$

$e^{(x^2)^2}(2x)$

$$\frac{d}{dx}\int_{0}^{5x}\frac{\sqrt{1+t^2}}{t}\,dt = \frac{\cancel{5}\sqrt{1+25x^2}}{\cancel{5}x}$$

Here are a couple with a little twist...

$$\frac{d}{dx}\int_{x}^{5} 3t\sin t\,dt = -3x\sin x$$

Lower bound is not a constant???

Compare these...

$$\int_{1}^{3} x^2\,dx = \frac{x^3}{3}\Big|_1^3$$

$$= \frac{27}{3} - \frac{1}{3}$$

$$= \frac{26}{3}$$

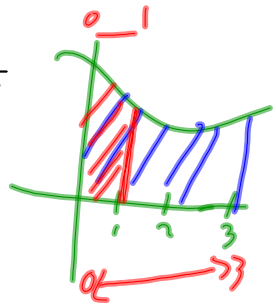
$$\int_{3}^{1} x^2\,dx = \frac{x^3}{3}\Big|_3^1$$

$$= \frac{1}{3} - \frac{27}{3}$$

$$= -\frac{26}{3}$$

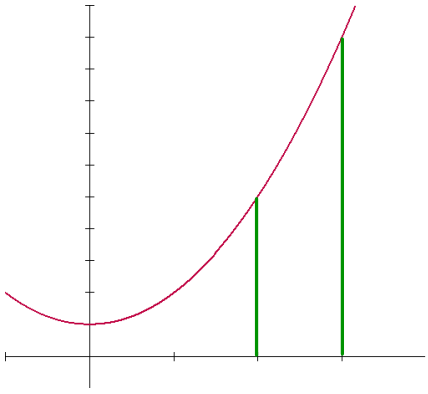
$$\frac{d}{dx}\int_{2x}^{x^2} \frac{1}{2+e^t}\,dt = \frac{2x}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$$

Neither bound is a constant???



Use this type of reasoning...

$$\int_{1}^{3} (x^2+1)\,dx = \int_{0}^{3} (x^2+1)\,dx - \int_{0}^{1} (x^2+1)\,dx$$



$$\frac{d}{dx}\int_{2x}^{x^2}\frac{1}{2+e^t}\,dt$$

$$= \frac{d}{dx}\int_{0}^{x^2}\frac{1}{2+e^t}\,dt - \int_{0}^{2x}\frac{1}{2+e^t}\,dt$$

$$= \frac{(2x)'}{2+e^{x^2}} - \frac{2}{2+e^{2x}}$$

Example:

Find $g(1)$, given that $g(x) = \frac{d}{dx}\int_{x^2}^{x^3}(3t - t^3)\,dt$ *

$$\frac{d}{dx}\int_0^{x^3}(3t-t^3)\,dt - \frac{d}{dx}\int_0^{x^2}(3t-t^3)\,dt$$

$$=(3x^3-x^9)(3x^2) - (3x^2-x^6)(2x)$$

$$g(1) = (3-1)(3)-(3-1)(2)$$

$$= 6-4$$

$$=2$$

Practice problems...

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#3, 7, 9, 11, 13, 15, 17, 21