Logarithmic Functions

Did You Know?

Logarithms were developed independently by John Napier (1550–1617), from Scotland, and Jobst Bürgi (1552–1632), from Switzerland. Since Napier published his work first, he is given the credit. Napier was also the first to use the decimal point in its modern context.

Logarithms were developed before exponents were used. It was not until the end of the seventeenth century that mathematicians recognized that logarithms are exponents.

$$\frac{d}{dx}\int_0^x \ln(t^2+1)\,dt$$

$$\boxed{0} = \ln(x^2+1)$$

Recall some knowledge about inverse functions...

•Inverse functions is the set of ordered pair obtained by interchanging the x and y values.

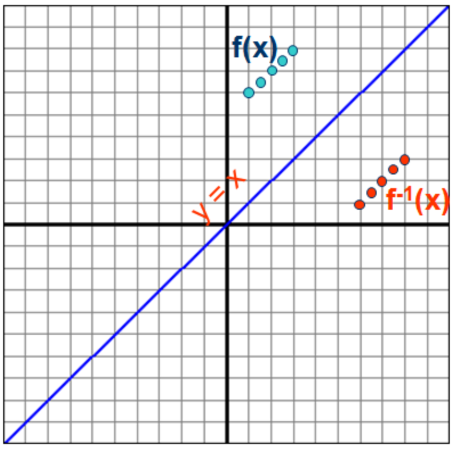
f(x)

x	y
2	12
3	13
4	14
5	15
6	16

$f^{-1}(x)$

x	y
12	2
13	3
14	4
:	5
	:

•Inverse functions can be created graphically by a reflection on the y = x axis.



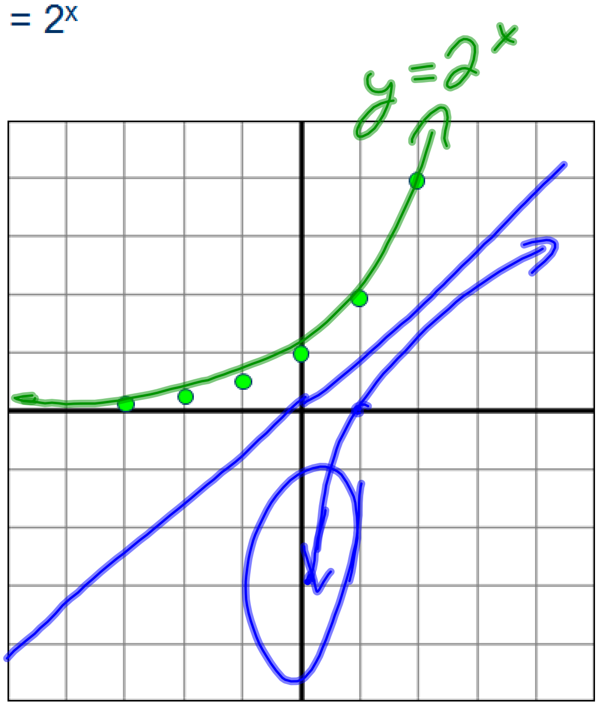
Word association game....

Addition ...	inverse process	...??
Multiplication ...	inverse process	...??
Squaring ...	inverse process	...??
Sine ...	inverse process	...??
Exponential ...	inverse process	...??

•Let us graph the exponential function $y = 2^x$

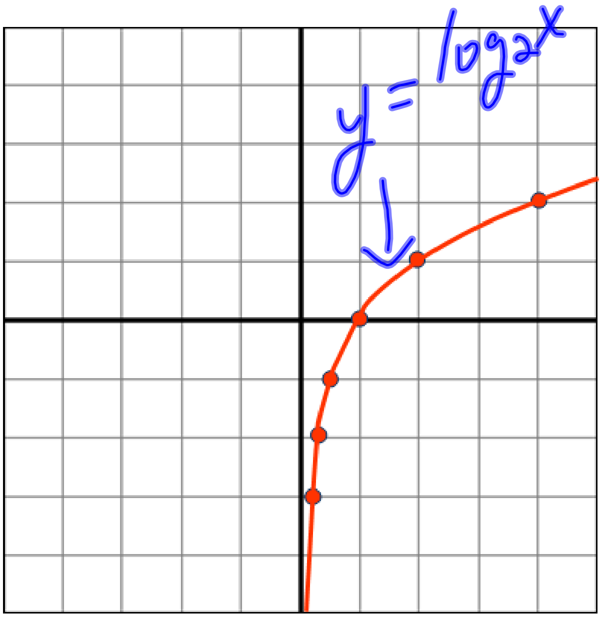
•Table of values:

x	y
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4



- Let us find the inverse the exponential function $y = 2^x$
- Table of values:

x	y
0.125	-3
0.25	-2
0.5	-1
1	0
2	1
4	2



- Next, you will find the inverse of an exponential algebraically
- Write the process in your notes

$y = a^x$

Interchange $x \rightarrow y$

$x = a^y$

- We write these functions as:

$x = a^y \longrightarrow y = \log_a x$

log 27

log2 7

base

$$x = a^y$$

exponent

$$y = \log_a x$$

exponent base

$\log x$ "Assumed Base 10"

ln

$\ln \Rightarrow \log_e$ "e" Euler's Number

IMPORTANT NOTATION!!!

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

Exponential Form		**Logarithmic Form**	
$x = a^y$	is written as…	$y = \log_a x$	← read as… "log of x to the base a"
Say, "the base a to the exponent y is x."		Say, "y is the exponent to which you raise the base a to get the answer x."	$a > 0$ and $a \neq 1$
			$x > 0$

Example 1) Write the following into logarithmic form:

$\log_{\square} \square = \square$

a) $3^3 = 27$ $\longrightarrow$ $\log_3 27 = 3$

b) $4^5 = 256$ $\longrightarrow$ $\log_4 256 = 5$

c) $2^7 = 128$ $\longrightarrow$ $\log_2 128 = 7$

d) $(1/3)^x = 27$ $\longrightarrow$ $\log_{1/3} 27 = x$

Example 2) Write the following into exponential form:

a) $\log_2 64=6$ ⟶ $2^6 = 64$

b) $\log_{25} 5=1/2$ ⟶ $25^{1/2} = 5$

c) $\log_8 1=0$ ⟶ $8^0 = 1$

d) $\log_{1/3} 9=2$ ⟶ $(1/3)^2 = 1/9$

$\log_7 18 = 1.485$

$7^1 = 7$
$7^2 = 49$

Translation: "7 Raised to what exponent equals 18?

$\log_6 36 = 2$

Example 3) Find the value of x for each example:

a) $\log_{1/3}27 = x$

$\left(\frac{1}{3}\right)^x = 27$

$(3^{-1})^x = 3^3$

$3^{-x} = 3^3$

$-x = 3$

$x = -3$

b) $\log_5 x = 3$

$5^3 = x$

$125 = x$

c) $\log_x(1/9) = 2$

$\sqrt{x^2} = \sqrt{\frac{1}{9}}$

$x = \frac{1}{3}$

d) $\log_3 x = 0$

$3^0 = x$

$1 = x$

What is $\log_{10} 1000$?

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

So now after reading the above...What is log 1000 ?

"3"

Review...

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

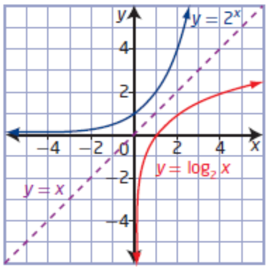
- a logarithm with base 10

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$
  - the range is $\{y \mid y \in \mathrm{R}\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

  $\log_{10} x = \log x$

Check-Up Time...

Evaluate each of the following:

1. $\log_2 8\sqrt{32} = x$

$\log_2 2^3(2^5)^{1/2}$

$\log_2 2^3(2^{5/2})$

$\log_2 2^{11/2}$

$= \frac{11}{2}$

$2^x = 8\sqrt{32}$

$2^x = 2^3(2^5)^{1/2}$

$2^x = 2^3(2^{5/2})$

$2^x = 2^{11/2}$

$x = \frac{11}{2}$

2. $-\frac{2}{3} = \log_x 81$

$\left(x^{-2/3}\right)^{-\frac{3}{2}} = 81^{-\frac{3}{2}}$

$x^1 = \frac{1}{(\sqrt{81})^3} = \frac{1}{729}$

3. $\log_5 \frac{1}{125}$

$\log_5 \frac{1}{5^3}$

$\log_5 5^{-3}$

$= -3$

4. $\log_{\sqrt{6}} 36 = x$

$\sqrt{6}^x = 36$

$6^{\frac{1}{2}x} = 6^2$

$\frac{1}{2}x = 2$

$x = 4$

What is the value of
any logarithm with an
argument of 1? Why?

$\log_{13} 1$

General Properties of Logarithms:

If $a > 0$ and $a \neq 1$, then…

(i) $\log_a 1 = 0$

(ii) $\log_a a^x = x$

(iii) $a^{\log_a x} = x$

$\log_7 7^5$

$7^? = 7^5$

$\log_m \sqrt{m} = \frac{1}{2}$

$\log_m m^{1/2} = \frac{1}{2}$

$m^? = m^{1/2}$

$$a^{\log_a x} = x$$

$$a^{\log_a x} = M \quad x$$

exponential

$\Downarrow$

logarithmic: $\log_a M = \log_a x$

$$\therefore M = x$$

$$3^{\log_3 7} = 7$$

$\log(a+b) \neq \log a + \log b$

Important!!!

logarithmic function does Not distribute!!

$7(x+3)$
$= 7x + 21$

14. $\log(4x-4)=2$

$10^2 = 4x-4$

$100+4=4x$

$\frac{104}{4}=\frac{4x}{4}$

$26 = x$

10. $\log_x 0.1=-1$

$(x^{-1})^{-1}=\left(\frac{1}{10}\right)^{-1}$

$x=10$

$\frac{1}{x}=\frac{1}{10}$

11. $\log_x 125=-3$

$(x^{-3})^{-\frac{1}{3}}=125^{-\frac{1}{3}}$

$x=\frac{1}{\sqrt[3]{125}}$

$x=\frac{1}{5}$

12. $\log_{\sqrt{3}} 27 = x$

$\sqrt{3}^{x}=27$

$(3^{\frac{1}{2}})^x=3^3$

$3^{\frac{1}{2}x}=3^3$

$(2)\frac{1}{2}x=3(2)$

$x=6$

13. $\log(x^2+9x)=1$

$10^1 = x^2+9x$

$x^2+9x-10=0$

$(x+10)(x-1)=0$

$x=-10 \quad x=1$

15. $\log_5 x = 3\log_5 7$

$5^{3\log_5 7} = x$

$\left(5^{\log_5 7}\right)^3 = x$

$(7)^3 = x$

① $b^{\log_b x} = x$

When finished....

Handout #1-5

Laws of Logarithms

Did You Know?

The world's first hand-held scientific calculator was the Hewlett-Packard HP-35, so called because it had 35 keys. Introduced in 1972, it retailed for approximately U.S. $395. Market research at the time warned that the demand for a pocket-sized calculator was too small. Hewlett-Packard estimated that they needed to sell 10 000 calculators in the first year to break even. They ended up selling 10 times that. By the time it was discontinued in 1975, sales of the HP-35 exceeded 300 000.

Laws of Logarithms: If *a > 0, M > 0, N > 0 and n ε R then…*

1) Product Law → the logarithm of a product is equal to the sum of the logarithms of the factors.

PROOF: Let $\log_a M = b$ and $\log_a N = c$

so $a^b = M$ $a^c = N$

then,

$$\log_a(MN) = \log_a(a^b \bullet a^c)$$
$$= \log_a(a^{b+c})$$
$$= b + c$$
$$\therefore \boxed{\log_a(MN) = \log_a M + \log_a N}$$

$a^b \cdot a^c$
$= a^{b+c}$

examples: a) $\log_{10}(6 \times 9)$

$= \log_{10} 6 + \log_{10} 9$

b) $\log_2 12 + \log_2 7$

$\log_2(12 \times 7)$

$\log_2(84)$

2) Quotient Law → the logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.

PROOF: Let $\log_a M = b$ and $\log_a N = c$

so $a^b = M$ and $a^c = N$

then,

$$\log_a\left(\frac{M}{N}\right) = \log_a\left(\frac{a^b}{a^c}\right)$$

$$= \log_a(a^{b-c})$$

$$= b - c$$

$$\therefore \boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N}$$

examples: a) $\log_5\left(\dfrac{97}{62}\right)$ b) $\log_2 15 - \log_2 3$ c) $\log_{10}\left(\dfrac{1}{9}\right)$

a) $= \log_5 97 - \log_5 62$

b) $\log_2\left(\frac{15}{3}\right)$

$\log_2 5$

c) $= \log 1 - \log 9$

$= 0 - \log 9$

$= \underline{-\log 9}$

3) Law of Logarithms for Powers → the logarithm of a power of a number is equal to the exponent multiplied by the logarithm of the number

PROOF: Let $\log_a M = b$

so $a^b = M$

then,

$$\log_a M^p = \log_a (a^b)^p$$
$$= \log_a (a^{b \times p})$$
$$= b \times p$$

$$\therefore \boxed{\log_a M^p = p \times \log_a M}$$

examples: a) $\log_{10} 8^9$

$= 9 \log 8$

b) $2\log_3 5$

$\log_3 5^2$

$\log_3 25$

c) $\log_5 \sqrt{125}$

$\log_5 125^{1/2}$

$\frac{1}{2} \log_5 125$

$= \frac{1}{2}(3)$

$= \frac{3}{2}$

Quiz:

1. a) $x = 9$

b) $x = 4, 1 \Rightarrow x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4, 1$

2/ $32(2^x)^2 = 12(2^x) - 1$

$m = 2^x$

$32m^2 = 12m - 1$

$\Downarrow$

$m = \frac{1}{8}$ $m = \frac{1}{4}$

$2^x = \frac{1}{8}$ $2^x = \frac{1}{4}$

$x = -3$ $x = -2$

3/

t	(2012) 0	1
P	17500	

$\times 0.988$

$t = 18$

Keep 98.8%

$$P = 17500(0.988)^t$$

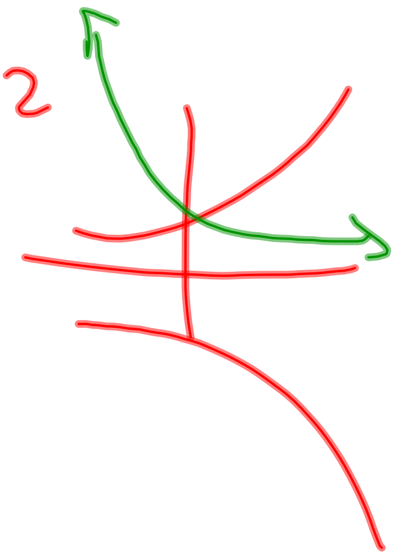
$$= 17500(0.988)^{18}$$

$$= 14082$$

4/ $\frac{2(y+4)}{2} = \frac{-18(2)^{3(x+2)}}{2} + \frac{24}{2}$

$y+4 = -9(2)^{3(x+2)} + 12$

$\boxed{y = -9(2)^{3(x+2)} + 8}$



$(x,y) \to \left(\frac{1}{3}x - 2, -9y + 8\right)$

$(0,1) \to \left(\frac{1}{3}(0) - 2, -9(1) + 8\right)$

$(-2,-1)$

5. $m = 50(0.9)^t$

90% Remains

(a) 10%

(b) $D = 50(0.9)^{28}$

$= 2.6\text{mg}$ Yes

6. (a) $P = 375(1.2)^{t/2}$

b) $P = 375(1.2)^{5/2}$

$= 592$ coyotes

$\frac{650\,C}{10\,800 \text{ Deer}}$

$592\,C \times \frac{10\,800\,D}{650\,C} = 9836$

7.

t	0	5750
m	4096	

$\times \frac{1}{2}$

$m = 4096\left(\frac{1}{2}\right)^{\frac{t}{5750}}$

$\frac{2}{4096} = \frac{4096}{4096}\left(\frac{1}{2}\right)^{\frac{t}{5750}}$

$\frac{1}{2048} = \left(\frac{1}{2}\right)^{\frac{t}{5750}}$

$\frac{1}{2^{11}} = \frac{1}{2^{t/5750}}$

$11 = \frac{t}{5750}$

$t = 63\,250$ years old

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_6 \frac{x}{y}$

b) $\log_5 \sqrt{xy}$

c) $\log_3 \frac{9}{\sqrt[3]{x^2}}$

d) $\log_7 \frac{x^5 y}{\sqrt{z}}$

a) $\log_6 x - \log_6 y$

b) $\log_5 (xy)^{1/2}$

$\frac{1}{2} \log_5 (xy)$

$\frac{1}{2} (\log_5 x + \log_5 y)$ ✓

b) $\log_5 x^{1/2} y^{1/2}$

$\log_5 x^{1/2} + \log_5 y^{1/2}$

$\frac{1}{2} \log_5 x + \frac{1}{2} \log_5 y$

c) $\log_3 \frac{9}{\sqrt[3]{x^2}}$

$= \log_3 9 - \log_3 x^{2/3}$

$= 2 - \frac{2}{3} \log_3 x$

d) $\log_7 \left(\frac{x^5 y}{\sqrt{z}} \right) \longrightarrow 5\log_7 x + \log_7 y - \frac{1}{2} \log_7 z$

$\log_7 (x^5 y) - \log_7 \sqrt{z}$

$\log_7 x^5 + \log_7 y - \log_7 z^{1/2}$

$5 \log_7 x + \log_7 y - \frac{1}{2} \log_7 z$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_3 9\sqrt{3}$

b) $\log_5 1000 - \log_5 4 - \log_5 2$

c) $2 \log_3 6 - \frac{1}{2} \log_3 64 + \log_3 2$

a) $\log_3 9 + \log_3 \sqrt{3}$

$\log_3 9 + \frac{1}{2} \log_3 3$

$= 2 + \frac{1}{2}(1)$

$= \frac{5}{2}$

b) $\log_5 \left(\frac{1000}{4}\right) - \log_5 2$

$\log_5 250 - \log_5 2$

$\log_5 \left(\frac{250}{2}\right)$

$\log_5 125$

$= 3$

b) $\log_5 \left(\frac{1000}{4(2)}\right)$

$\log_5 (125)$

$= 3$

$$2 \log_3 6 - \frac{1}{2} \log_3 64 + \log_3 2$$

$\log_3 36 - \log_3 8 + \log_3 2$

$\log_3 \left(\frac{36 \times 2}{8}\right)$

$\log_3 9$

$= 2$

$\log_3 \left(\frac{36}{8}\right) + \log_3 2$

$\log_3 \left(\frac{36}{8} \times 2\right)$

$\log_3 9 = 2$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $4 \log_3 x - \frac{1}{2}(\log_3 x + 5 \log_3 x)$

b) $\log_2 (x^2 - 9) - \log_2 (x^2 - x - 6)$

$\log_2\left(\frac{x^2-9}{x^2-x-6}\right)$

$\log_2 \frac{(\cancel{x-3})(x+3)}{(\cancel{x-3})(x+2)}$

$\log_2\left(\frac{x+3}{x+2}\right)$

a) $\frac{8}{2}\frac{4}{1}\log_3 x - \frac{1}{2}\log_3 x - \frac{5}{2}\log_3 x$

$= 1 \log_3 x$

1

2) $\log_3 x^4 - \frac{1}{2}(\log_3 x + \log_3 x^5)$

$\log_3 x^4 - \frac{1}{2}(\log_3(x \cdot x^5))$

$\log_3 x^4 - \frac{1}{2}\log_3 x^6$

$\log_3 x^4 - \log_3 (x^6)^{\frac{1}{2}}$

$\log_3 x^4 - \log_3 x^3$

$\log_3\left(\frac{x^4}{x^3}\right)$

$\log_3 x$

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12\left(\log_b x^2 - 2\log_b x\right) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$

a) Express the following as a single logarithm… $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

$\log_2 3^4 + \log_2 6 - \log_2 3^3$

$\log_2\left(\frac{3^4 \cdot 6}{3^3}\right) = \log_2 18$

b) Evaluate the following… $\log_2 (32)^{\frac{1}{3}}$

$\log_N N = N$

$\frac{1}{3}\log_2 32$

$\frac{1}{3}(5)$

$= \frac{5}{3}$

$\log_2 (2^5)^{1/3}$

$\log_2 2^{5/3} \Rightarrow \frac{5}{3}\log_2 2$

$= \frac{5}{3}$

$\frac{5}{3}(1)$

$= \frac{5}{3}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

$\frac{1}{2}\left[\log_5 ab^2 - \log_5 c^3\right]$

$\frac{1}{2}\log_5\left(\frac{ab^2}{c^3}\right)$

$\log_5\left(\frac{ab^2}{c^3}\right)^{1/2} = \log_5\sqrt{\frac{ab^2}{c^3}}$

Systematic Algorithm:

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

① Remove all Brackets $\Rightarrow \frac{1}{2}\log_5 a + \log_5 b - \frac{3}{2}\log_5 c$

② Return all coefficients as exponents $\Rightarrow \log_5 a^{1/2} + \log_5 b - \log_5 c^{3/2}$

③ Write as a single logarithm $\Rightarrow \log_5\left(\frac{a^{1/2}b}{c^{3/2}}\right)$

$\log_5\left(a^{1/2}b\,c^{-3/2}\right)$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12\left(\log_b x^2 - 2\log_b x\right) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$

$\frac{3}{4}\left(12\log_b x^2 - 24\log_b x + 8\log_b x^{1/2} - 4\log_b x^{-7}\right)$

$9\log_b (x^2)^9 - 18\log_b x + 6\log_b x^{1/2} - 3\log_b x^{-7}$

$\log_b x^{18} - \log_b x^{18} + \log_b x^3 - \log_b x^{-21}$

$\log_b\left(\frac{\cancel{x^{18}} \cdot x^3}{\cancel{x^{18}} \cdot x^{-21}}\right)$

$= \log_b x^{24}$

$= 24\log_b x$

① $\log_2 x^{-205}$ ② 22 ③ $\frac{625}{16}$