Hint!!!

Sometimes completing the square can be a useful strategy to evaluate an integral...

$$\int \frac{1}{x^2+4x+13}\,dx$$

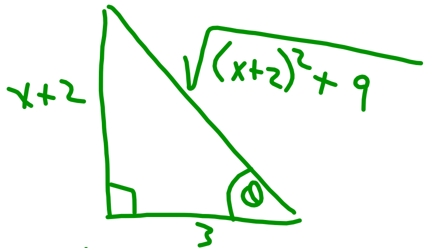
$x^2+4x+13$

$(x^2+4x+4)+13-4$

$(x+2)^2+9$

$$\int \frac{dx}{(x+2)^2+9}$$

$$\int \frac{3\cancel{\sec^2\theta}\,d\theta}{9\cancel{\sec^2\theta}}$$



$$\frac{1}{3}\int d\theta$$

$$=\frac{1}{3}\theta+C$$

$$=\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+C$$

$3\sec\theta=\sqrt{(x+2)^2+9}$

$9\sec^2\theta=(x+2)^2+9$

$3\tan\theta=x+2$

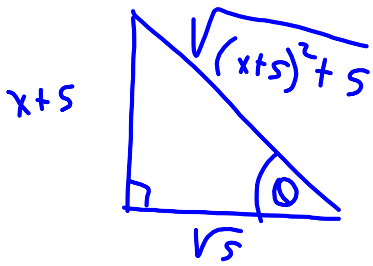
$3\sec^2\theta\,d\theta=dx$

$$\int \frac{dx}{x^2+10x+30}$$

$(x^2+10x+25)+30-25$

$(x+5)^2+5$

$$\int \frac{dx}{(x+5)^2+5}$$



$$\int \frac{\sqrt{5}\cancel{\sec^2\theta}\,d\theta}{5\cancel{\sec^2\theta}}$$

$$\frac{\sqrt{5}}{5}\int d\theta$$

$$\frac{\sqrt{5}}{5}\theta+C$$

$$\frac{\sqrt{5}}{5}\tan^{-1}\left(\frac{x+5}{\sqrt{5}}\right)+C$$

$\sqrt{5}\sec\theta=\sqrt{(x+5)^2+5}$

$5\sec^2\theta=(x+5)^2+5$

$\sqrt{5}\tan\theta=x+5$

$\sqrt{5}\sec^2\theta\,d\theta=dx$

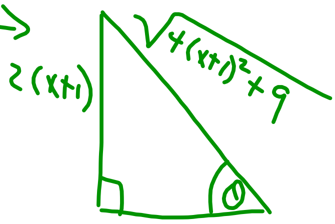
$$\int \frac{dx}{\left(4x^2+8x+13\right)^2}$$

This will require us to complete the square on the denominator in order to set up a trigonometric substitution

$4(x^2+2x+1)+13-4$

$4(x+1)^2+9$

$$\int \frac{dx}{\left(4(x+1)^2+9\right)^2}$$



$$\int \frac{\frac{3}{2}\sec^2\theta\, d\theta}{81\sec^4\theta}$$

$\frac{\cancel{3}}{2}\times\frac{1}{\cancel{81}_{27}}=\frac{1}{54}$

$(3\sec\theta)^4=\left(\sqrt{4(x+1)^2+9}\right)^4$

$81\sec^4\theta=\left(4(x+1)^2+9\right)^2$

$$\frac{1}{54}\int\frac{1}{\sec^2\theta}\,d\theta$$

$$\frac{1}{54}\int\cos^2\theta\, d\theta$$

$\tan\theta=\frac{2(x+1)}{3}$

$3\tan\theta=2x+2$

$\frac{3\sec^2\theta\, d\theta}{2}=\frac{\cancel{2}dx}{\cancel{2}}$

Double Angle Identity

$$\frac{1}{54}\int\left(\frac{\cos 2\theta+1}{2}\right)d\theta$$

$\cos 2\theta=2\cos^2\theta-1$

$\frac{\cos 2\theta+1}{2}=\cos^2\theta$

$$\frac{1}{108}\int\left(\cos 2\theta+1\right)d\theta$$

$$\frac{1}{108}\left[\frac{1}{2}\sin 2\theta+\theta\right]+C$$

$$\frac{1}{216}\left(2\sin\theta\cos\theta\right)+\frac{\theta}{108}+C$$

$$\frac{1}{108}\left(\frac{2(x+1)}{\sqrt{4(x+1)^2+9}}\right)\left(\frac{3}{\sqrt{4(x+1)^2+9}}\right)+\frac{1}{108}\tan^{-1}\left(\frac{2(x+1)}{3}\right)+C$$

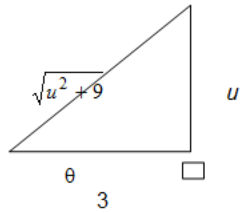
$$\frac{6(x+1)}{108\left(4(x+1)^2+9\right)}+\frac{1}{108}\tan^{-1}\left(\frac{2(x+1)}{3}\right)+C$$

Solution:

$$\int \frac{dx}{(4x^2+8x+13)^2} = \int \frac{dx}{\left(4\left(x^2+2x+1\right)+13-4\right)^2}$$

$$= \int \frac{dx}{\left(4\,(x+1)^2+9\right)^2} \qquad u^2 = 4\,(x+1)^2 \quad u = 2\,(x+1) \quad du = 2\,dx$$

$$= \frac{1}{2}\int \frac{du}{\left(u^2+9\right)^2}$$



$$\tan\theta = \frac{u}{3}, \qquad 3\tan\theta = u, \qquad 3\sec^2\theta\, d\theta = du,$$

$$\frac{\sqrt{u^2+9}}{3} = \sec\theta,\ \sqrt{u^2+9} = 3\sec\theta$$

$$\int \frac{dx}{(4x^2+8x+13)^2} = \frac{1}{2}\int \frac{du}{\left(u^2+9\right)^2}$$

$$= \frac{1}{2}\int \frac{3\sec^2\theta\, d\theta}{(3\sec\theta)^4}$$

$$= \frac{1}{2}\int \frac{3\sec^2\theta\, d\theta}{81\sec^4\theta}$$

$$= \frac{1}{54}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{54}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{54}\int \frac{(1+\cos 2\theta)d\theta}{2}$$

$$= \frac{1}{108}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{108}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

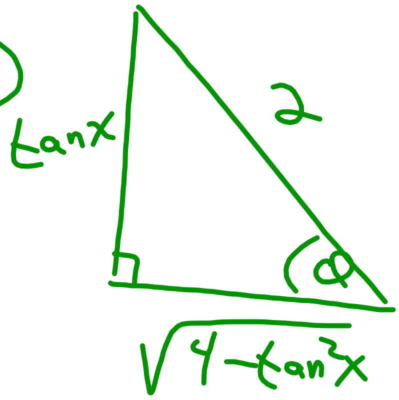
$$= \frac{1}{108}\left(\tan^{-1}\frac{u}{3} + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{u}{3} + \frac{u}{\sqrt{u^2+9}}\,\frac{3}{\sqrt{u^2+9}}\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{u}{3} + \frac{3u}{u^2+9}\right) + C$$

$$= \frac{1}{108}\left(\tan^{-1}\frac{2\,(x+1)}{3} + \frac{6\,(x+1)}{4\,(x+1)^2+9}\right) + C$$

23/ $$\int \frac{\sec^2 x \, dx}{(4-\tan^2 x)^{3/2}}$$



$$\int \frac{2\cos\theta \, d\theta}{8\cos^3\theta}$$

$2\sin\theta = \tan x$

$2\cos\theta \, d\theta = \sec^2 x \, dx$

$$\frac{1}{4}\int \left(\frac{1}{\cos^2\theta}\right) d\theta$$

$$\frac{1}{4}\int \sec^2\theta \, d\theta$$

$$\frac{1}{4}\tan\theta + C$$

$$\frac{1}{4}\left(\frac{\tan x}{\sqrt{4-\tan^2 x}}\right) + C$$

$2\cos\theta = \sqrt{4-\tan^2 x}$

$8\cos^3\theta = (4-\tan^2 x)^{3/2}$

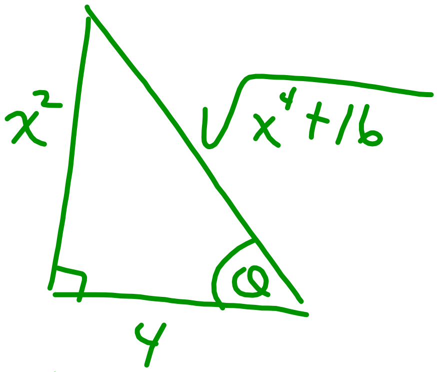
#3/ $\int \frac{x}{x^4+16}\,dx$

$\int \frac{2\sec^2\theta\,d\theta}{16\sec^2\theta}$

$\frac{1}{8}\int d\theta$

$\frac{1}{8}\theta + C$

$\frac{1}{8}\tan^{-1}\left(\frac{x^2}{4}\right) + C$



$4\sec\theta = \sqrt{x^4+16}$

$16\sec^2\theta\, d\theta = x^4+16$

$\tan\theta = \frac{x^2}{4}$

$4\tan\theta = x^2$

$4\sec^2\theta\, d\theta = 2x\,dx$

$2\sec^2\theta\, d\theta = \underline{\underline{x\,dx}}$

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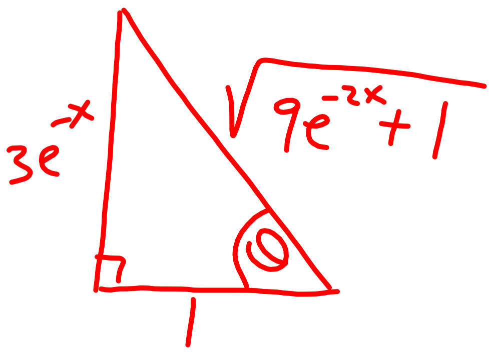

$$\sec\theta\left(\frac{1}{\tan\theta}\right)$$

$$\frac{1}{\cos\theta}\ \cot\theta$$

$$\frac{1}{\cancel{\cos\theta}}\left(\frac{\cancel{\cos\theta}}{\sin\theta}\right)$$

$$\int\left(\frac{1}{\sin\theta}\right)d\theta$$

$$\int \csc\theta$$