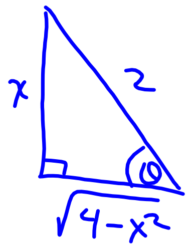
Warm Up

Evaluate each of the following:

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{\sqrt{3}-1}{4}$$



$2\sin\theta = x$

$2\cos\theta\, d\theta = dx$

$2\cos\theta = \sqrt{4-x^2}$

$$\int \frac{\cancel{2\cos\theta}\, d\theta}{(2\sin\theta)^2 \cancel{2\cos\theta}}$$

$$-\frac{1}{4}\int \frac{(-)1}{\sin^2\theta}$$

$$-\frac{1}{4}\int -\csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta$$

$$= -\frac{1}{4}\left(\frac{\sqrt{4-x^2}}{x}\right)\Bigg|_1^{\sqrt{2}} = -\frac{1}{4}\left[\left(\frac{\sqrt{2}}{\sqrt{2}} - \sqrt{3}\right)\right]$$

$$= -\frac{1}{4}(1-\sqrt{3})$$

$$\int \frac{2x+4}{x^3-2x^2}\,dx = -2\ln|x| + \frac{2}{x} + 2\ln|x-2| + C$$

Hint:

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} = \frac{2x+4}{x^2(x-2)}$$

$$\frac{Ax(x-2)+B(x-2)+Cx^2}{x^2(x-2)} = \frac{2x+4}{x^2(x-2)}$$

$$Ax^2 - 2Ax + Bx - 2B + Cx^2 = 2x+4$$

$A + C = 0$ $\quad -2A + B = 2$ $\quad -2B = 4$

$-2 + C = 0$ $\quad -2A + (-2) = 2$ $\quad B = -2$

$C = 2$ $\quad -2A = 4$

$A = -2$

$$-2\int \frac{dx}{x} + -2\int x^{-2}\,dx + 2\int \frac{dx}{x-2}$$

$$-2\ln|x| + 2x^{-1} + 2\ln|x-2| + C$$

Differential Equations

Differential Equation:

- An equation that involves one or more derivatives of an unknown function.

Example: $\frac{dy}{dx} = x^2$

* This means that the unknown is a function and not a number

The solutions of the equation $\frac{dy}{dx} = x^2$ are...

$$y = \int x^2 \, dx$$

$$= \frac{x^3}{3} + C$$

The order of a differential equation is the order of the highest derivative that it contains...

$$\frac{dy}{dx} = 3y \quad (1^{\text{st}} \ order)$$

$$y'' + y' = \cos t \quad (2^{\text{nd}} \ order)$$

First-Order Differential Equations

A function $y = y(x)$ is a ***solution*** of a differential equation on a given interval if the equation is satisfied for every x in that interval when y and its derivatives are substituted in the equation.

For example, $y = e^{2x}$ is a solution of the differential equation $\frac{dy}{dx} - y = e^{2x}$

$y = e^{2x}$

$y' = e^{2x}(2)$

Verify:

LS	RS
$2e^{2x} - 1e^{2x}$	e^{2x}
e^{2x}	

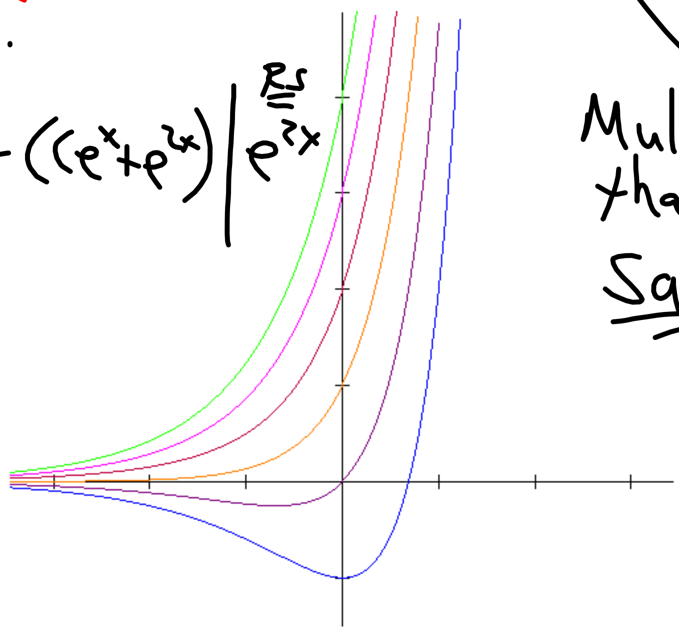
LS = RS

However, this is not the only solution; for example $y = Ce^{x} + e^{2x}$ is also a solution for every real value of C.

$\frac{dy}{dx} - y = e^{2x}$

There is a family of curves that correspond to the possible solutions.

LS	RS
$((e^{x} + e^{2x}(2)) - ((e^{x} + e^{2x})$	e^{2x}
$= e^{2x}$	



Multiple functions that satisfy same equation

Initial Value Problem provided with an initial condition.

$$\frac{dy}{dx} - y = e^{2x}, \quad \boxed{y(0) = 3}$$

(Must solve for C)

We know the general solution is... $y = Ce^{x} + e^{2x}$

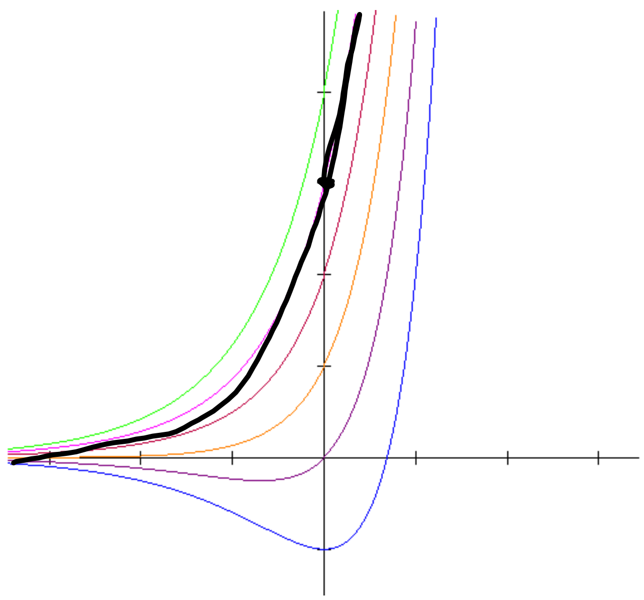
$y(0) = 3$ implies that when x = 0, y = 3

$x=0 \quad y=3$

$y = Ce^{x} + e^{2x}$

$$\left.\begin{array}{l} 3 = Ce^{0} + e^{0} \\ 3 = C + 1 \\ 2 = C \end{array}\right\} \quad y = 2e^{x} + e^{2x}$$

The graph of this solution would be the integral curve that passes through the point (0, 3).



xample 1 Show that $y(x)=x^{-\frac{3}{2}}$ is a solution to $4x^2y''+12xy'+3y=0$ for $x>0$.

$$y'=-\frac{3}{2}x^{-\frac{5}{2}}$$

$$y''=\frac{15}{4}x^{-\frac{7}{2}}$$

$$\underline{\underline{LS}}$$

$$4x^2\left(\frac{15}{4}x^{-\frac{7}{2}}\right)+\cancel{12}^{6}x\left(-\frac{3}{\cancel{2}}x^{-\frac{5}{2}}\right)+3\left(x^{-\frac{3}{2}}\right)$$

$$15x^{-\frac{3}{2}}-18x^{-3/2}+3x^{-3/2}$$

$$0$$

$$\underline{RS}$$

$$0$$

$$\underline{LS=RS}$$

Warm Up

2. (a) Show that

$$y = c_1 \sin 2x + 3\cos 2x$$

is a general solution for the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

(b) Show that $\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$ has a solution of $y = c_1 + c_2e^{2x}$

$$y = c_1 \sin 2x + 3\cos 2x$$

(a) $y' = c_1\cos 2x(2) - 3\sin 2x(2) = 2c_1\cos 2x - 6\sin 2x$

$y'' = -2c_1\sin(2x)(2) - 6\cos 2x(2) = -4c_1\sin 2x - 12\cos 2x$

Verify:

LS | RS
$(-4c_1\sin 2x - 12\cos 2x) + 4(c_1\sin 2x + 3\cos 2x)$ | 0
$-4c_1\sin 2x - 12\cos 2x + 4c_1\sin 2x + 12\cos 2x$
0

(b) Show that $\frac{d^2y}{dx^2} = 2\frac{dy}{dx}$ has a solution of $y = c_1 + c_2e^{2x}$

$y' = c_2e^{2x}(2) = 2c_2e^{2x}$

$y'' = 2c_2e^{2x}(2) = 4c_2e^{2x}$

$\underline{LS}$	$\underline{RS}$
$4c_2e^{2x}$	$2(2c_2e^{2x})$
	$4c_2e^{2x}$

$\underline{LS = RS}$

First-Order Separable Equations

- A separable differential equation can be written in the form

$$\frac{dy}{dx} = g(x)\,f(y)$$

- To solve this equation we write it in the differential form

$$h(y) = g(x)dx, \text{ where } h(y) = 1/f(y).$$

- In this form the variables have been separated:
  - All the y's are on one side of the equation, and
  - all the x's are on the other.

Example:

- Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

- Find the solution of the equation that satisfies the initial condition $y(0) = 2$.

$x=0$
$y=2$

$\boxed{y(x) = ?}$

$(y^2)\cancel{(dx)}\frac{dy}{\cancel{dx}} = \frac{x^2}{y^2}(dx)(y^2)$

$\int y^2\,dy = \int x^2\,dx$

$\frac{y^3}{3} + C = \frac{x^3}{3}$ (+C on Oneside only)

Solve for "C": $x=0 \quad y=2$

$\frac{8}{3} + C = 0$

$C = -\frac{8}{3}$

Isolate for $y=$

$\frac{y^3}{3} - \frac{8}{3} = \frac{x^3}{3}$

$y^3 - 8 = x^3$

$y^3 = x^3 + 8$

$\boxed{y = \sqrt[3]{x^3+8}}$

~~$y = x+2$~~

Family of curves in solution...

(0,2)

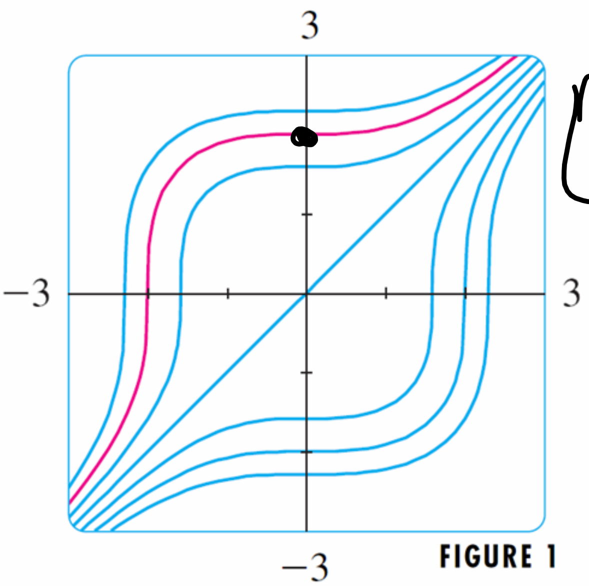


FIGURE 1

Example:

Solve the differential equation $\frac{dy}{dx} = -4xy^2$

and then the initial-value problem

$\frac{dy}{dx} = -4xy^2$, $y(0) = 1$ $\qquad y(x) = ?$

$\int \frac{1}{y^2} dy = \int -4x \, dx$

$-y^{-1} + C = -2x^2$

$-\left(\frac{1}{1}\right) + C = -2(0)^2 \quad \leftarrow$ When $x = 0$, $y = 1$

$-1 + C = 0$

$C = 1$

$-\frac{1}{y} + 1 = -2x^2$

$-\frac{1}{y} = -2x^2 - 1$

$\frac{1}{y} = \frac{2x^2+1}{1}$

$y = \frac{1}{2x^2+1}$

Example:

Solve the initial value problem

$$(4y-\cos y)\frac{dy}{dx}-3x^2=0\ ,\quad y(0)=0 \qquad 2y^2-\sin y=x^3$$

$$\int(4y-\cos y)\,dy=\int 3x^2\,dx$$

$$2y^2-\sin y+C=x^3$$

$$2(0)^2-\sin(0)+C=0$$

$$0-0+C=0$$

$$C=0$$

$$\boxed{2y^2-\sin y=x^3}$$

ex. $$\frac{d}{dx}\int_{x^2}^{3x}(2t^3-3)\,dt$$

$$\frac{d}{dx}\int_0^{3x}(2t^3-3)\,dt-\frac{d}{dx}\int_0^{x^2}(2t^3-3)\,dt$$

$$=\left[2(3x)^3-3\right](3)-\left[2(x^2)^3-3\right](2x)$$

$$=3(54x^3-3)-2x(2x^6-3)$$

$$\int_2^5$$

$$\int_0^5-\int_0^2=\int_2^5$$

$$\int \sin^3 x \cos^5 x \, dx$$

$$\int \underline{\sin x} \sin^2 x \cos^5 x \, dx$$

$$\int \sin x (1 - \cos^2 x) \cos^5 x \, dx$$

$$-\int \cos^5 x \, {-\sin x} \, dx + \int \cos^7 x \, {-\sin x} \, dx$$

$$-\frac{1}{6} \cos^6 x + \frac{1}{8} \cos^8 x + C$$

$\sin^7 x$ ✓

$(\sin x)^7$ ✓

$\sin x^7$ ✗

$$\int \frac{e^{5x}}{2-e^{5x}}\,dx$$

$u = 2 - e^{5x}$

$du = -e^{5x}(5)\,dx$

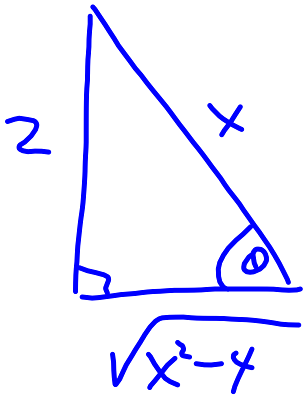
$-\frac{1}{5}du = e^{5x}\,dx$

$$-\frac{1}{5}\int \frac{du}{u}$$

$= -\frac{1}{5}\ln u + C$

$= -\frac{1}{5}\ln(2 - e^{5x}) + C$

$$\int \frac{dx}{x^2\sqrt{x^2-4}}$$



$$2\cot\theta=\sqrt{x^2-4}$$

$$2\csc\theta=x$$

$$-2\csc\theta\cot\theta\,d\theta=dx$$

$$\int \frac{-\cancel{2}\csc\theta\,\cancel{\cot\theta}\,d\theta}{(4\csc^2\theta)(\cancel{2\cot\theta})}$$

$$-\frac{1}{4}\int\frac{1}{\csc\theta}\,d\theta$$

$$-\frac{1}{4}\int\sin\theta\,d\theta$$

$$=\frac{1}{4}\cos\theta+C$$

$$=\frac{1}{4}\left(\frac{\sqrt{x^2-4}}{x}\right)+C$$

$$\int x^3 \ln x \, dx$$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^4}{4}$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int \frac{x^4}{x} dx$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 dx$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{16} x^4 + C$$

$$\int \frac{1}{\sin x - 1}\,dx$$

$$\boxed{\sin^2 x + \cos^2 x = 1}$$

$$\sin^2 x - 1 = -\cos^2 x$$

$$\int \frac{1}{\sin x - 1}\left(\frac{\sin x + 1}{\sin x + 1}\right)dx$$

$$\int \frac{\sin x + 1}{\sin^2 x - 1}\,dx$$

$$\int \frac{(\sin x + 1)}{-\cos^2 x}\,dx$$

$$-\int (\cos x)^{-2}(\sin x + 1)\,dx$$

$$-\int \left((\cos x)^{-2}\sin x + (\cos x)^{-2}\right)dx$$

$$-\int (\cos x)^{-2}\sin x\,dx - \int \frac{1}{\cos^2 x}\,dx$$

$$+\int (\cos x)^{-2}(-\sin x)\,dx - \int \sec^2 x\,dx$$

$$-1(\cos x)^{-1} - \tan x + C$$

$$\int \frac{\sqrt{\ln x}}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int u^{1/2}\,du$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(\ln x)^{3/2} + C$$

$$\int \tan^{-1} 3x \, dx$$

$u = \tan^{-1} 3x \qquad dv = dx$

$du = \dfrac{3}{9x^2+1} dx \qquad v = x$

$$= x\tan^{-1}3x - \int x\left(\frac{3}{9x^2+1}\right)dx$$

$$= x\tan^{-1}3x - \frac{3}{18}\int \frac{18x}{9x^2+1}\,dx$$

$$= x\tan^{-1}3x - \frac{1}{6}\ln(9x^2+1) + C$$

Attachments

Review - Practice Test for Sinusoidal Functions.doc

Review - Trigonometric Functions(3)(4).doc