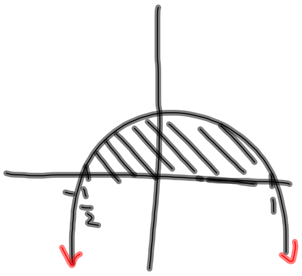
ex. $f(x) = -2x^2 + x + 1$ $\quad A = \frac{9}{8}u^2$

Find area bound by $f(x)$ and the x-axis.




$-2x^2+x+1=0$

$2x^2-x-1=0$

$2x^2-2x+x-1=0$

$2x(x-1)+1(x-1)=0$

$(2x+1)(x-1)=0$

$x=-\frac{1}{2},\ 1$

$\Delta x = \frac{3}{2} \times \frac{1}{n} = \frac{3}{2n}$ $\qquad x_k = -\frac{1}{2} + \frac{3k}{2n}$

$$A = \frac{3}{2n}\sum_{k=1}^{n} f\left(-\frac{1}{2}+\frac{3k}{2n}\right)$$

$$A = \frac{3}{2n}\sum_{k=1}^{n}\left[2\left(-\frac{1}{2}+\frac{3k}{2n}\right)^2 + \left(-\frac{1}{2}+\frac{3k}{2n}\right)+1\right]$$

$$A = \frac{3}{2n}\left[\sum_{k=1}^{n}\left[-2\left(\frac{1}{4}-\frac{6k}{4n}+\frac{9k^2}{4n^2}\right)-\frac{1}{2}+\frac{3k}{2n}+1\right]\right]$$

$$A = \frac{3}{2n}\sum_{k=1}^{n}\left(\cancel{-\frac{1}{2}}+\frac{3k}{n}-\frac{9k^2}{2n^2}\cancel{-\frac{1}{2}}+\frac{3k}{2n}\cancel{+1}\right)$$

$$A = \frac{3}{2n}\sum_{k=1}^{n}\left(-\frac{9k^2}{2n^2}+\frac{3k}{n}+\frac{3k}{2n}\right)$$

$$A = \frac{3}{2n}\left[-\frac{9}{2n^2}\sum_{k=1}^{n}k^2+\frac{3}{n}\sum_{k=1}^{n}k+\frac{3}{2n}\sum_{k=1}^{n}k\right]$$

$$A = \frac{3}{2n}\left[-\frac{9}{2n^2}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)+\frac{3}{n}\left(\frac{n^2}{2}+\frac{n}{2}\right)+\frac{3}{2n}\left(\frac{n^2}{2}+\frac{n}{2}\right)\right]$$

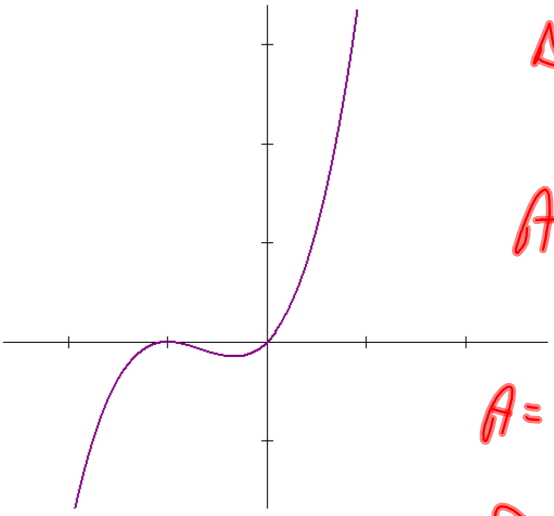
$$A = \frac{3}{2n}\left[-\frac{3n}{2}-\frac{9}{4}-\frac{9}{12n}+\frac{3n}{2}+\frac{3}{2}+\frac{3n}{4}+\frac{3}{4}\right]$$

$$A = \lim_{n\to\infty} -\frac{9}{4}\ \underbrace{-\frac{27}{8n}}_{\to 0}\ \underbrace{-\frac{27}{24n^2}}_{\to 0}+\frac{9}{4}\ \underbrace{+\frac{9}{4n}}_{\to 0}+\frac{9}{8}\ \underbrace{+\frac{9}{8n}}_{\to 0}$$

$$A = \frac{9}{8}u^2$$

Homework question...

Area below $f(x) = x^3 + 2x^2 + x$ between $x = 0$ and $x = 1$.



$\Delta x = \frac{1}{n} \qquad x_k = \frac{k}{n}$

$$A = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

$$A = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k^3}{n^3} + 2\left(\frac{k^2}{n^2}\right) + \frac{k}{n}\right)$$

$$A = \frac{1}{n}\left[\frac{1}{n^3}\sum_{k=1}^{n} k^3 + \frac{2}{n^2}\sum_{k=1}^{n} k^2 + \frac{1}{n}\sum_{k=1}^{n} k\right]$$

$$A = \frac{1}{n}\left[\frac{1}{n^3}\left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}\right) + \frac{2}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + \frac{1}{n}\left(\frac{n^2}{2} + \frac{n}{2}\right)\right]$$

$$A = \frac{1}{n}\left[\frac{n}{4} + \frac{1}{2} + \frac{1}{4n} + \frac{2n}{3} + 1 + \frac{2}{6n} + \frac{n}{2} + \frac{1}{2}\right]$$

$$\lim_{n\to\infty} A = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} + \frac{2}{3} + \frac{1}{n} + \frac{2}{6n^2} + \frac{1}{2} + \frac{1}{2n}$$

$$= \frac{1}{4} + \frac{2}{3} + \frac{1}{2}$$

$$= \frac{3 + 8 + 6}{12}$$

$$= \frac{17}{12} u^2$$

# Definite Integral

## 2 Definition of a Definite Integral

integral symbol

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

width of rectangles

### Remarks

- In the notation $\int_a^b f(x)dx$...
  - the symbol $\int$ is called an *integral sign* and resembles a stretched-out "S";
  - $f(x)$ is called the *integrand* and $a$ and $b$ are the *upper* and *lower limits of integration*, resp.
  - we could replace $x$ with any other letter without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

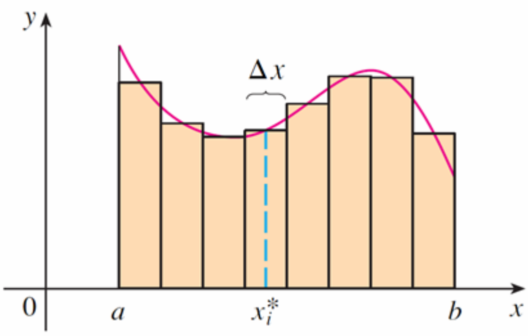



**FIGURE 1**
If $f(x) \geqslant 0$, the Riemann sum $\Sigma\, f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

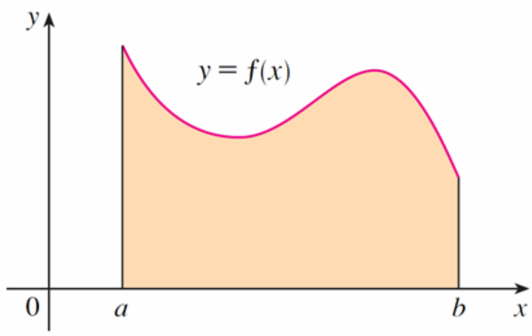



**FIGURE 2**
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from $a$ to $b$.

What if a function takes on both positive and negative values?

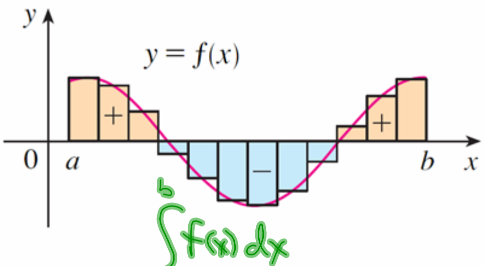



**FIGURE 3**
$\Sigma f(x_i^*)\,\Delta x$ is an approximation to the net area

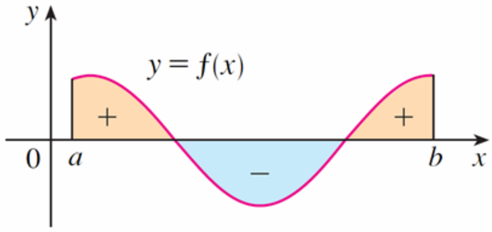



**FIGURE 4**
$\int_a^b f(x)\,dx$ is the net area

Example: $$\int_0^3 \left(x^3 - 6x\right)dx$$

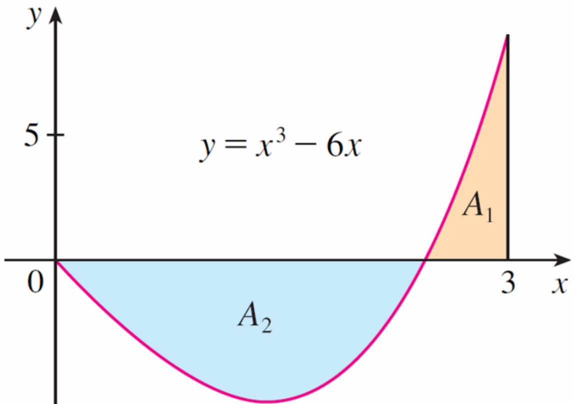



**FIGURE 6**

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

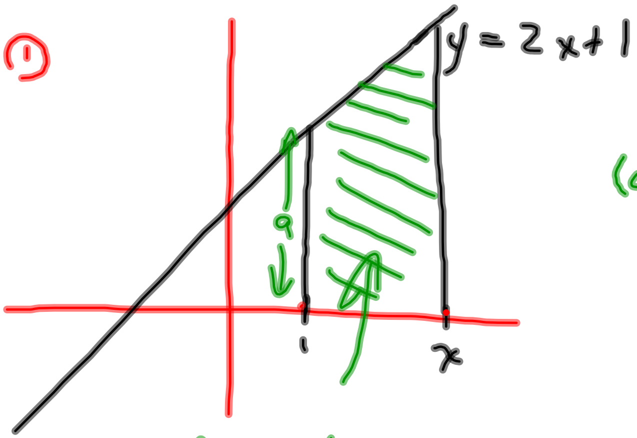
$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

1




(a) Find an area function

$A = \frac{1}{2}(a+b)(h)$

$A = \frac{1}{2}(3+2x+1)(x-1)$

$A = \frac{1}{2}(4+2x)(x-1)$

$A = (2+x)(x-1)$

$A = 2x-2+x^2-x$

$A = x^2+x-2$ ← Antiderivative of f(x)

$A' = 2x+1$

upper  lower boundary

$F(x) - F(1)$

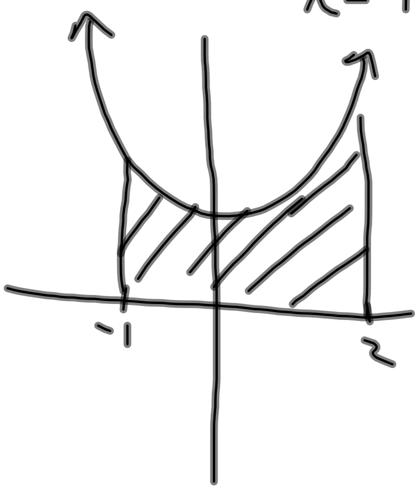
$F(x) = x^2+x$

$F(1) = 2$

$A = (x^2+x) - 2$

$A = x^2+x-2$

$y = x^2 + 1$, find area between $x = -1$ and $x = 2$.




$F(x) = \frac{x^3}{3} + x$

$A = F(2) - F(-1)$

$\left(\frac{8}{3} + 2\right) - \left(\frac{-1}{3} + 1\right)$

$= \frac{9}{3} + \frac{3}{1}$

$= \frac{18}{3} = \underline{6u^2}$

## Evaluation Theorem

**Evaluation Theorem** If $f$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$, that is, $F' = f$.

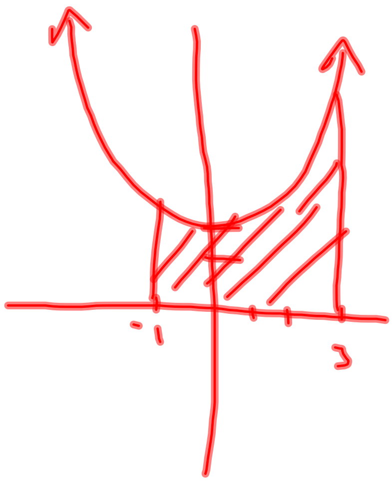
$F(x) = \frac{x^3}{3}$

$$\int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3} \cdot 1^3 - \tfrac{1}{3} \cdot 0^3 = \tfrac{1}{3}$$

Much easier than using a Riemann sum !!!

Example:

Determine the area below the curve $f(x) = 3x^2 + 2$ between $x$ = -1 and $x = 3$.




$$\int_{-1}^{3} (3x^2+2)\,dx$$

$$= x^3 + 2x \Big|_{-1}^{3}$$

$$= (27+6) - (-1-2)$$

$$= 33 - (-3)$$

$$= 36\,u^2$$