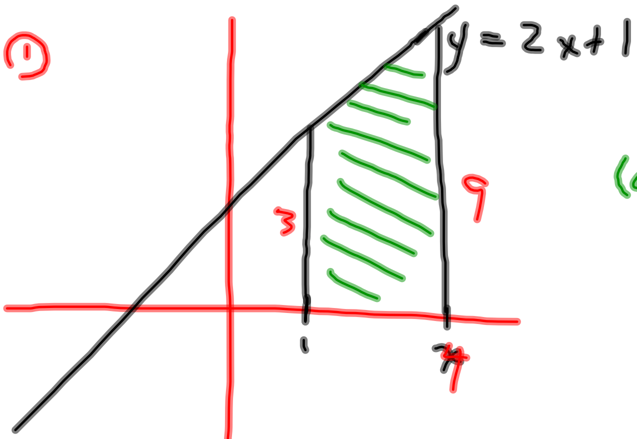
1



(a) Find an area function

$$\int_1^4 (2x+1)\,dx$$

$$= x^2 + x \Big|_1^4$$

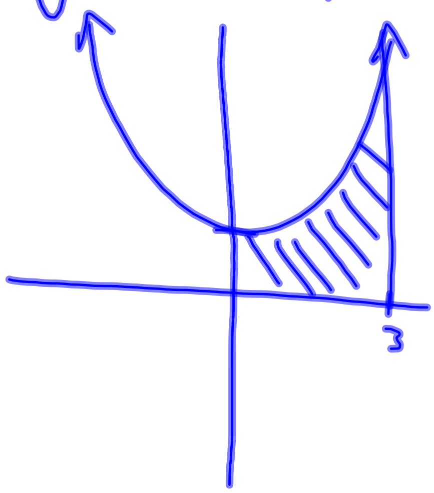
$$= \left[(4)^2+(4)\right] - \left[(1)^2+(1)\right]$$

$$= \underline{18u^2}$$

$A = \frac{1}{2}(3+9)(3)$

$A = \frac{1}{2}(36) = \underline{18u^2}$

$y = x^2 + 1$, Area between $x = 0$ and 3.



$\int f(x)\,dx$

$$\int_0^3 (x^2+1)\,dx$$

$$= \frac{x^3}{3} + x \cancel{+C} \Big|_0^3$$

$$= \left(\frac{3^3}{3} + 3 \cancel{+C}\right) - \left(\frac{0}{3} + 0 \cancel{+C}\right)$$

$$= 9 + 3 = \underline{12 u^2}$$

ex.

Determine the area bound by $f(x)=-2x^2+x+1$ and the x-axis using a Riemann Sum.

$0=-2x^2+x+1$

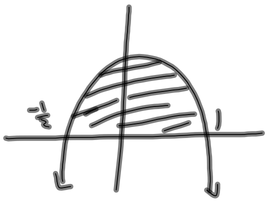
$0=2x^2-x-1$

$0=2x^2-2x+x-1$

$0=2x(x-1)+1(x-1)$

$0=(2x+1)(x-1)$

$x=\frac{-1}{2},1$



$\Delta x=\frac{3}{2n}$ $\quad x_k=\frac{-1}{2}+\frac{3k}{2n}$

$f(x)=-2x^2+x+1$

$$A=\frac{3}{2n}\sum_{k=1}^{n} f\left(-\frac{1}{2}+\frac{3k}{2n}\right)$$

$$A=\frac{3}{2n}\sum_{k=1}^{n}\left[-2\left(-\frac{1}{2}+\frac{3k}{2n}\right)^2+\left(\frac{-1}{2}+\frac{3k}{2n}\right)+1\right]$$

$$A=\frac{3}{2n}\sum_{k=1}^{n}\left[-2\left(\frac{1}{4}-\frac{6k}{4n}+\frac{9k^2}{4n^2}\right)-\frac{1}{2}+\frac{3k}{2n}+1\right]$$

$$A=\frac{3}{2n}\sum_{k=1}^{n}\cancel{-\frac{1}{2}}+\frac{6k}{2n}-\frac{9k^2}{2n^2}\cancel{-\frac{1}{2}}+\frac{3k}{2n}\cancel{+1}$$

$$A=\frac{3}{2n}\sum_{k=1}^{n}\left(\frac{9k}{2n}-\frac{9k^2}{2n^2}\right)$$

$$A=\frac{3}{2n}\left[\frac{9}{2n}\sum_{k=1}^{n}k-\frac{9}{2n^2}\sum_{k=1}^{n}k^2\right]$$

$$A=\frac{3}{2n}\left[\frac{9}{2n}\left(\frac{n^2}{2}+\frac{n}{2}\right)-\frac{9}{2n^2}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\right]$$

$$A=\frac{3}{2n}\left(\frac{9n}{4}\cancel{+\frac{9}{4}}-\frac{9n}{6}\cancel{-\frac{9}{4}}-\frac{9}{12n}\right)$$

$$A=\frac{27}{8}-\frac{27}{12}-\frac{27}{24n^2}$$

$$\lim_{n\to\infty}\left(\frac{27}{8}-\frac{27}{12}-\frac{27}{24n^2}\right)\quad \frac{27}{24n^2}\to 0$$

$$A=\frac{27}{8}-\frac{27}{12}$$

$$A=\frac{81-54}{24}$$

$$A=\frac{27}{24}=\frac{9}{8}u^2$$

Using Integration

$$\int_{-\frac{1}{2}}^{1}(-2x^2+x+1)\,dx$$

$$=-\frac{2}{3}x^3+\frac{x^2}{2}+x\Big|_{-\frac{1}{2}}^{1}$$

$$=\left(-\frac{2}{3}+\frac{1}{2}+1\right)-\left(\frac{-2}{3}\left(\frac{-1}{8}\right)+\frac{1}{8}-\frac{1}{2}\right)$$

$$=\left(-\frac{2}{3}+\frac{3}{2}\right)-\left(\frac{1}{12}+\frac{1}{8}-\frac{1}{2}\right)$$

$$=-\frac{2}{3}+2-\frac{1}{12}-\frac{1}{8}$$

$$=\frac{-16+48-2-3}{24}$$

$$=\frac{27}{24}=\frac{9}{8}$$

$$\int \left(3x^7 - x^{-8} + \tfrac{1}{7}\sec^2 7x\,(7)\right)dx$$ ← Indefinite Integral... (No Bounds)

Integrate... (Antidifferentiate)

$$= \frac{3}{8}x^8 + \frac{1}{7}x^{-7} + \frac{1}{7}\tan 7x + C$$

Evaluation Theorem

Evaluation Theorem If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, $F' = f$.

$$\int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3} \cdot 1^3 - \tfrac{1}{3} \cdot 0^3 = \tfrac{1}{3}$$

Much easier than using a Riemann sum !!!

Example:

Determine the area below the curve $f(x) = 3x^2 + 2$ between $x = -1$ and $x = 3$.

$$\int_{-1}^{3} (3x^2+2)\,dx$$

$$= x^3 + 2x \Big|_{-1}^{3}$$

$$= (27+6) - (-1-2)$$

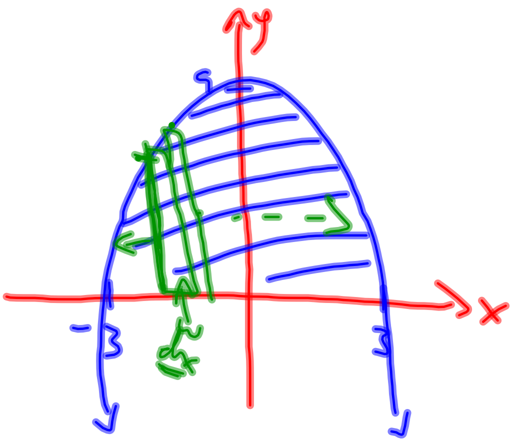
$$= 33 - (-3)$$

$$= \underline{36u^2}$$

Example:

Determine the area bound by the curve $y = 9 - x^2$ and the x-axis.

$A =$



$$\int_{-3}^{3} \underbrace{(9-x^2)}_{\text{height}}\, \underbrace{dx}_{\text{width}}$$

$$= 9x - \frac{x^3}{3} \Big|_{-3}^{3}$$

$$= (27-9) - (-27+9)$$

$$= \underline{36u^2}$$

OR $2\int_{0}^{3} (9-x^2)\,dx$

Indefinite Integrals

- Because of the relation between antiderivatives and integrals, the notation $\int f(x)dx$ is traditionally used for an antiderivative of f and is called an *indefinite integral*.
- Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

Example:

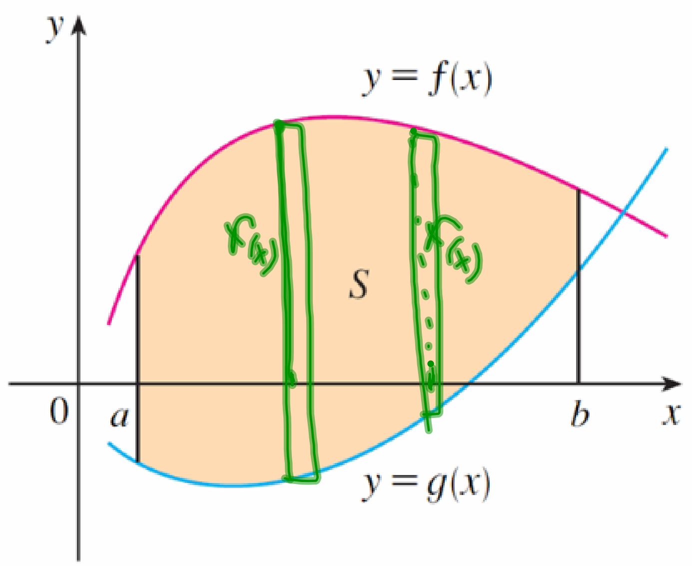
$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C$$

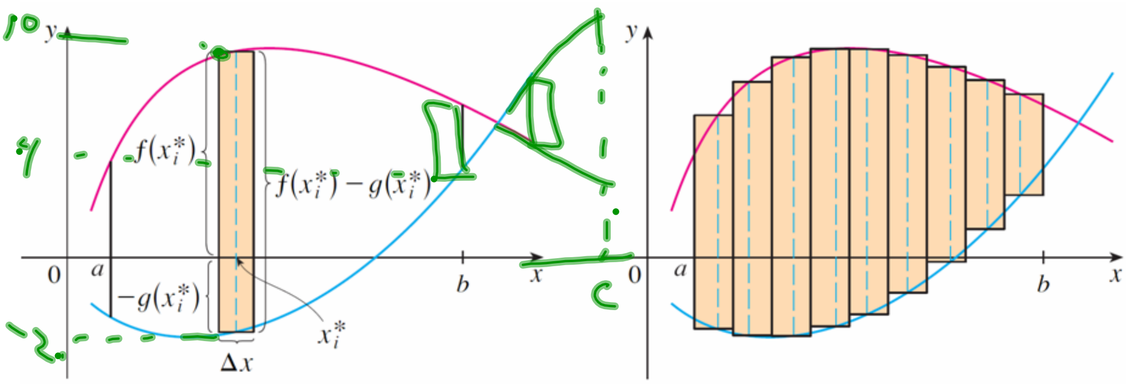
$$= 2x^5 - 2\tan x + C$$

Area Between Curves

What if we would like to determine the area between two curves?



Could we use rectangular strips?



(a) Typical rectangle **FIGURE 2** (b) Approximating rectangles

Write out a definite integral that would represent the area of region S.

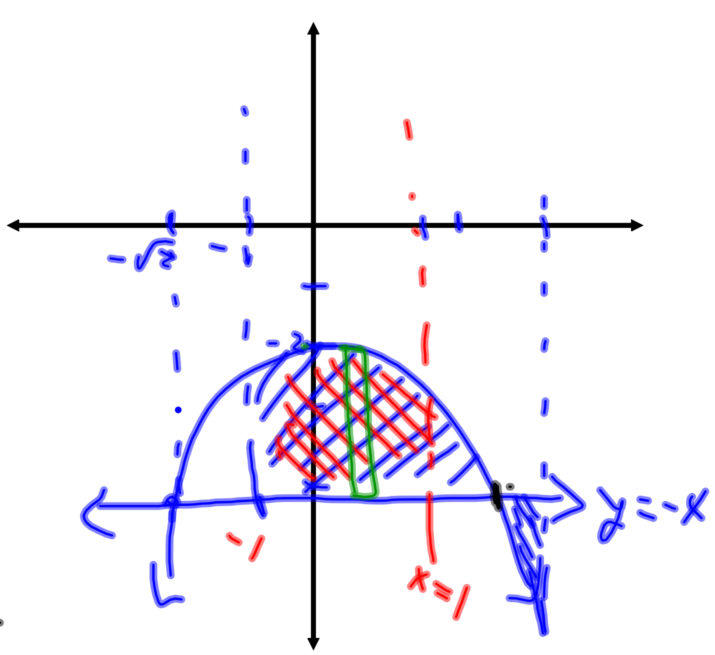
$$\int_a^b [f(x)-g(x)]\,dx$$

Example:

Determine the area bounded by the curve $f(x) = -x^2 - 2$ and the lines $x = -1$, $x = 2$ and $y = -4$.

$x=1$

1. Start with a sketch...



$-x^2-2=-4$

$x^2-2=0$

$(x-\sqrt{2})(x+\sqrt{2})=0$

$x=\pm\sqrt{2}$

$$\int_{-1}^{1}[(-x^2-2)-(-4)]dx$$

$$=\int_{-1}^{1}(-x^2+2)dx$$

$$=-\frac{x^3}{3}+2x\Big|_{-1}^{1}$$

$$=\left(-\frac{1}{3}+2\right)-\left(\frac{1}{3}-2\right)$$

$$=\frac{4}{1}-\frac{2}{3}$$

$$=\frac{10}{3}u^2$$

$$\int_{-1}^{\sqrt{2}}(-x^2-2)-(-4)\,dx$$

$$+\int_{\sqrt{2}}^{2}[-4-(-x^2-2)]dx$$

Example:

Determine the area bounded by the curves $f(x) = -x^2 + 9$ and $g(x) = 2x^2 - 3$.

Points of Intersection

$-x^2+9 = 2x^2-3$

$-3x^2+12=0$

$3x^2-12=0$

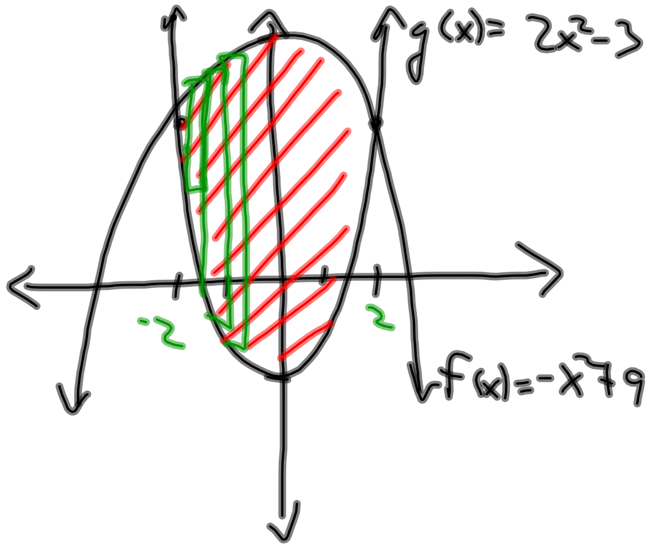
$3(x^2-4)=0$

$3(x-2)(x+2)=0$

$x=\pm 2$

$f(2) = -4+9$
$= 5$

$f(-2) = -4+9$
$=5$



$$\int_{-2}^{2} \left[(-x^2+9)-(2x^2-3)\right]dx$$

$$\int_{-2}^{2} (-3x^2+12)\,dx$$

$$= -x^3+12x \Big|_{-2}^{2}$$

$$=(-8+24)-(8-24)$$

$$= 16-(-16)$$

$$= 32u^2$$

Determine area enclosed by

$f(x) = x^2 - 2x - 8$ and $y = -2x + 1$

Pts. of intersection...

$x^2 - 2x - 8 = -2x + 1$

$(x-4)(x+2) = 0$

$x = 4, -2$

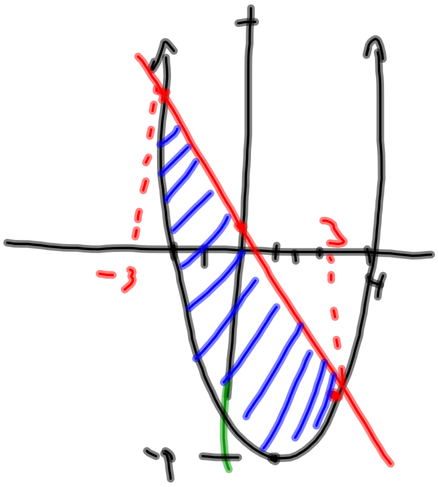
$x^2 - 9 = 0$

$(x-3)(x+3) = 0$

$x = \pm 3$

$y = (x^2 - 2x + 1) - 8 - 1$

$y = (x-1)^2 - 9$



$$\int_{-3}^{3} \left[(-2x+1) - (x^2 - 2x - 8)\right] dx$$

$$\int_{-3}^{3} (-x^2 + 9)\, dx$$

$$= -\frac{x^3}{3} + 9x \Big|_{-3}^{3}$$

$$= (-9 + 27) - (9 - 27)$$

$$= 18 - (-18)$$

$$= \underline{36u^2}$$

Practice
Sheet
#1,2,3