Warm Up

Differentiate the following:

$$f(x)=\frac{-2x\tan^{-1}\sqrt{x}}{\cos^{-1}\left(\sec x^{3}\right)}$$

$$f'(x)=\frac{\left[(-2)\tan^{-1}\sqrt{x}+(-2x)\left(\frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right)\right)\right]\cos^{-1}(\sec x^3)-\\ -2x\tan^{-1}\sqrt{x}\left(-\frac{1}{\sqrt{1-(\sec x^3)^2}}\sec x^3\tan x^3(3x^2)\right]}{\left[\cos^{-1}(\sec x^3)\right]^2}$$

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#4) $f(x)=\left(3\tan^{-1}x\right)^4$ $\qquad f'(\sqrt{3})=?$

$$f'(x)=4\left(3\tan^{-1}x\right)^3\left(3\,\frac{1}{1+x^2}\right)$$

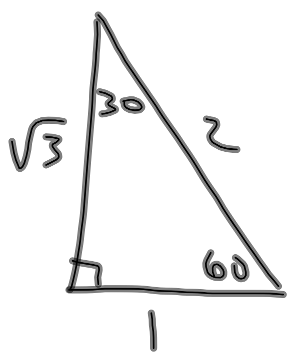
$$f'(\sqrt{3})=4\left(3\tan^{-1}(\sqrt{3})\right)^3 3\left(\frac{1}{1+3}\right)$$

$$=\left(\frac{12}{4}\right)27\left(\tan^{-1}\sqrt{3}\right)^3$$

$$=81\left(\tan^{-1}\sqrt{3}\right)^3$$

$$=81\left(\frac{\pi}{3}\right)^3$$

$$=\frac{81\pi^3}{27}$$



$$=3\pi^3$$

6) $f(x) = (x-3)\sqrt{6x-x^2} + 9\sin^{-1}\left(\frac{x-3}{3}\right)$ $f'(3)$

$f'(x) = (1)\sqrt{6x-x^2} + (x-3)\left[\frac{1}{2}(6x-x^2)^{-\frac{1}{2}}(6-2x)\right] +$

$9\left(\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}\right)\frac{1}{3}(1)$

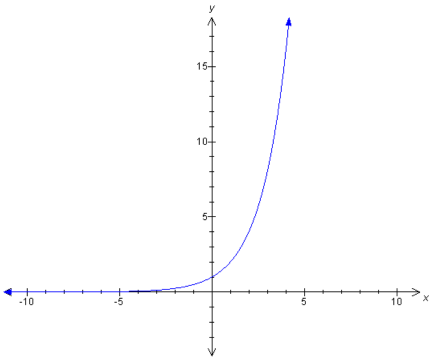
$= 3 + 0 + 9\left(\frac{1}{3}\right)$

$= 6$

Differentiating Exponential Functions

What is an exponential function?

$y = 2^x$ ← Base must be a Real number



When you do not have a rule to differentiate resort to the definition...

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's try and differentiate $y = a^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

This factor does not depend on *h*, therefore we can move to the front of the limit

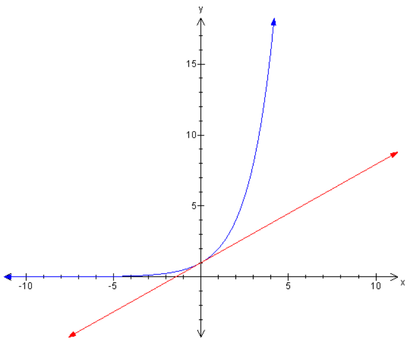
Thus we now have...

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

What would be the value of $f(0)$?

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

What would this represent in terms of slope??



We have determined that $f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$

Same thing!

and that $\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "*e*"...Euler's Number

```
e^(1)
        2.718281828...
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This leads to the following definition...

Definition of the Number e

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

What does this mean geometrically?

- Geometrically, this means that
  - of all the exponential functions $y = a^x$,
  - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 .

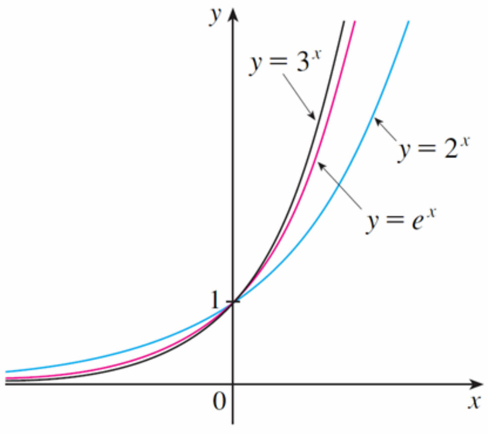

FIGURE 6

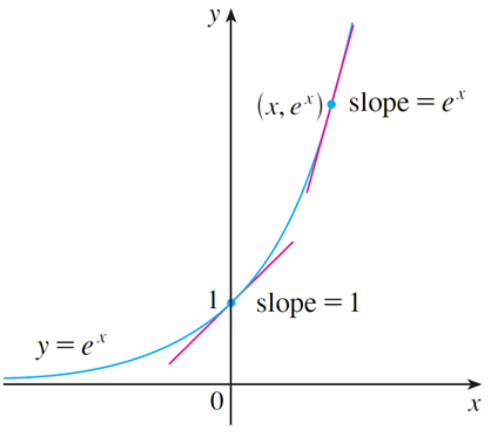

FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

This is the ONLY function that is its own derivative

$$f(x) = e^x$$
$$f'(x) = e^x$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$

ex. $y = e^{5x^7}$

$y' = e^{5x^7}(35x^6)$

Practice Exercises

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#4, 5, 6, 8, 9, 10,

Derivatives of Logarithmic Functions

Let's work from the known...

- At this point you should know how to differentiate $y = e^x$.

What other function could this model?

$$\ln y = x$$

$\log_e y = x$

Natural logarithm → $\ln$

Try to differentiate ~~$y = \ln x$.~~

(Implicit)

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

Differentiate: $y = \ln x^3$

$e^y = x^3$

$e^y \frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{e^y} = \frac{3x^2}{x^3}$

$d(\ln 10x^7) = \frac{70x^6}{10x^7}$

$d(\ln u) = \frac{du}{u}$

Rule: $d(\ln u) = \frac{1}{u} du$

We have now covered base "e"...both as an exponential and logarithmic function...

What about other bases??

Will need to know the change of base formula for logarithms:

$$\log_N M = \frac{\log_b M}{\log_b N}$$

Whatever new base you choose

Differentiate:

$$y = \log_6 x^3$$

$$y = \frac{\ln x^3}{\ln 6}$$

$$y = \frac{1}{\ln 6} \ln x^3$$

$$y' = \frac{1}{\ln 6}\left(\frac{3x^2}{x^3}\right)$$

$$y = \log(5x^4)$$

$$y = \frac{\ln 5x^4}{\ln 10}$$

$$y = \frac{1}{\ln 10} \ln 5x^4$$

$$y' = \frac{1}{\ln 10}\left(\frac{20x^3}{5x^4}\right)$$

$$y = \log_b u$$

$$y' = \frac{du}{u \ln b}$$

Rule: $d\left(\log_b u\right) = \dfrac{1}{u \ln b}\, du$

This leaves one form of exponential function remaining...

- What about a function such as $y = 3^{9x}$

$\ln y = \ln 3^{9x}$

$\frac{1}{y}\frac{dy}{dx} = 9x \ln 3$

$\frac{dy}{dx} = (9 \ln 3)(y)$

$\frac{dy}{dx} = 3^{9x} \ln 3 (9)$

$\log_3 y = 9x$

$\frac{1}{y \ln 3}\frac{dy}{dx} = 9$

$\frac{dy}{dx} = y \ln 3 (9)$

$\frac{dy}{dx} = 3^{9x} \ln 3 (9)$

$d(b^u) = b^u \ln b \, du$

Try this one... $y = \pi^{x^5}$

$y' = \pi^{x^5} \ln \pi (5x^4)$

Rule:

$$d(b^u) = b^u(\ln b)du \quad , where \;\; b \in R$$

Practice Problems:

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#1 #2 a #3 #4

#5 #6 #7 #8