Solutions to Worksheet for Sections 4.6
Integration by Substitution

Math S-1ab
Calculus I and II

July 18, 2007

Find the following integrals. In the case of an indefinite integral, your answer should be the most general antiderivative. In the case of a definite integral, your answer should be a number.

In these problems, a substitution is given.

1. $\int (3x-5)^{17}\,dx$, $u = 3x-5$

Solution. As suggested, we let $u = 3x - 5$. Then $du = 3\,dx$ so $dx = \frac{1}{3}\,du$. Thus

$$\int (3x-5)^{17}\,dx = \frac{1}{3}\int u^{17}\,du = \frac{1}{3}\cdot\frac{1}{18}u^{18} + C = \frac{1}{54}(3x-5)^{18} + C$$

□

2. $\int_0^4 x\sqrt{x^2+9}\,dx$, $u = x^2+9$

Solution. We have $du = 2x\,dx$, which takes care of the x and dx that appear in the integrand. The limits are changed to $u(0) = 9$ and $u(4) = 25$. Thus

$$\begin{aligned}\int_0^4 x\sqrt{x^2+9}\,dx &= \frac{1}{2}\int_9^{25}\sqrt{u}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_9^{25} \\ &= \frac{1}{3}\cdot(125-27) = \frac{98}{3}.\end{aligned}$$

□

3. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$, $u = \sqrt{x}$.

Solution. We have $du = \frac{1}{2\sqrt{x}}\,dx$, which means that

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C$$

□

4. $\displaystyle\int \frac{\cos 3x\,dx}{5+2\sin 3x}$, $u = 5 + 2\sin 3x$

Solution. We have $du = 6\cos 3x\,dx$, so

$$\int \frac{\cos 3x\,dx}{5+2\sin 3x} = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|5+2\sin 3x| + C$$

□

In these problems, you need to determine the substitution yourself.

5. $\displaystyle\int (4-3x)^7\,dx.$

Solution. Let $u = 4 - 3x$, so $du = -3x\,dx$ and $dx = -\frac{1}{3}du$. Thus

$$\int (4-3x)^7\,dx = -\frac{1}{3}\int u^7\,du = -\frac{u^8}{24} + C = -\frac{1}{24}(4-3x)^8 + C$$

□

6. $\displaystyle\int_{\pi/4}^{\pi/3} \csc^2(5x)\,dx$

Solution. Let $u = 5x$, so $du = 5\,dx$ and $dx = \frac{1}{5}du$. So

$$\begin{aligned}\int_{\pi/4}^{\pi/3} \csc^2(5x)\,dx &= \frac{1}{5}\int_{5\pi/4}^{5\pi/3} \csc^2 u\,du \\ &= -\frac{1}{5}\cot u\Big|_{5\pi/4}^{5\pi/3} \\ &= \frac{1}{5}\left[\cot\left(\frac{5\pi}{4}\right) - \cot\left(\frac{5\pi}{3}\right)\right] \\ &= \frac{1}{5}\left[1 + \frac{\sqrt{3}}{3}\right]\end{aligned}$$

□

7. $\displaystyle\int x^2 e^{3x^3-1}\,dx$

Solution. Let $u = 3x^3 - 1$, so $du = 9x^2\,dx$ and $dx = \frac{1}{9}du$. So

$$\begin{aligned}\int x^2 e^{3x^3-1}\,dx &= \frac{1}{9}\int e^u\,du = \frac{1}{9}e^u + C \\ &= \frac{1}{9}e^{3x^3-1}.\end{aligned}$$

□

Sometimes there is more than one way to skin a cat:

8. *Find* $\int \frac{x}{1+x}\,dx$*, both by long division and by substituting* $u = 1 + x$*.*

Solution. Long division yields

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}$$

So

$$\int \frac{x}{1+x}\,dx = x - \int \frac{dx}{1+x}$$

To find the leftover integral, let $u = 1 + x$. Then $du = dx$ and so

$$\int \frac{dx}{1+x} = \int \frac{du}{u} = \ln|u| + C$$

Therefore

$$\int \frac{x}{1+x}\,dx = x - \ln|x+1| + C$$

Making the substitution immediately gives $du = dx$ and $x = u - 1$. So

$$\begin{aligned}\int \frac{x}{1+x}\,dx = \int \frac{u-1}{u}\,du &= \int \left(1 - \frac{1}{u}\right)\,du \\ &= u - \ln|u| + C \\ &= x + 1 - \ln|x+1| + C\end{aligned}$$

It may appear that the two solutions are "different." But the difference is a constant, and we know that antiderivatives are only unique up to addition of a constant. □

9. *Find* $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$*, both by substituting* $u = z^2 + 1$ *and* $u = \sqrt[3]{z^2+1}$*.*

Solution. In the first substitution, $du = 2z\,dz$ and the integral becomes

$$\int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du = \frac{3}{2}u^{2/3} + C = \frac{3}{2}(z^2+1)^{2/3}$$

In the second, $u^3 = z^2 + 1$ and $3u^2\,du = 2z\,dz$. The integral becomes

$$\int \frac{3u^2}{u}\,du = \int 3u^2\,du = \frac{3}{2}u^3 + C = \frac{3}{2}(z^2+1)^{2/3} + C.$$

□The second one is a dirtier substitution, but the integration is cleaner.

Use the trigonometric identity

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

to find

10. $\int \sin^2 x\,dx$

Solution. Using $\cos 2\alpha = 1 - 2\sin^2\alpha$, we get

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$$

So

$$\begin{aligned}\int \sin^2 x\,dx &= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx \\ &= \frac{x}{2} - \frac{1}{4}\sin 2x + C.\end{aligned}$$

□

11. $\int \cos^2 x\,dx$

Solution. Using $\cos 2\alpha = 2\cos^2\alpha - 1$, we get

$$\sin^2\alpha = \frac{1+\cos 2\alpha}{2}$$

So

$$\begin{aligned}\int \sin^2 x\,dx &= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx \\ &= \frac{x}{2} + \frac{1}{4}\sin 2x + C.\end{aligned}$$

□

12. *Find*

$$\int \sec x \, dx$$

by multiplying the numerator and denominator by $\sec x + \tan x$*.*

Solution. We have

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} \, dx \end{aligned}$$

Now notice the numerator is the derivative of the denominator. So the substitution $u = \tan x + \sec x$ gives

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C.$$

□