Exponential Growth and Decay

Law of Exponential Change

Exponential growth problems involve growth in which the rate of change is proportional to the amount present.

ie. The more bacteria in a petri dish, the faster they multiply
The more radioactive material present, the faster it decays
The more money in your bank account, the faster it grows (assuming compound interest)

The differential equation that describes this type of growth is…

$\frac{dy}{dt} = ky \quad (k > 0)$, where k is the growth constant (if positive) or the decay constant (if negative).

This differential equation can be solved by separating the variables:

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}dy = kdt$$ ← Separate Variables

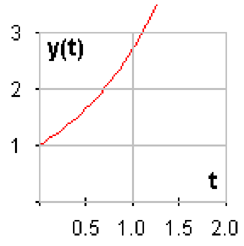
$$\ln|y| = kt + C$$ ← Antidifferentiate both sides

$$|y| = e^{kt+C}$$ ← Convert to exponential form

$$|y| = e^{C}e^{kt}$$ ← Laws of exponents

$$\frac{dy}{dt} = ky \quad y = \pm e^{C}e^{kt}$$ ← Definition of absolute value

$$\boxed{y = Ae^{kt}}$$ ← Let $A = \pm e^{C}$

INITIAL VALUE PROBLEM	SOLUTION	GRAPH
$\frac{dy}{dt} = ky$ $y(0) = y_o$	$y(t) = y_o e^{kt}$	

Notice that f(0)=A , so this implies that the constant A is the size of the quantity at t = 0.

Example:

The population of bacteria grown in a culture follows the law of natural growth, with a growth rate of 15% per hour. If there are 10 000 bacteria present initially, how many will there be after 4 hours?

Let P denote the number of bacteria present at time t :

$$\frac{dP}{dt} = \boxed{k}P \Rightarrow P = Ae^{kt}$$

$$\frac{dp}{dt} = 0.15P$$

$$P = Ae^{0.15t} \text{ ,when t } = 0, \text{P} = 10\,000, \text{so}$$

$$10\ 000 = Ae^{0} \text{ ,hence A} = 10\,000$$

$$\therefore P = 10\,000e^{0.15t}$$

$$substitute\ t = 4\ldots$$

$$P = 10000e^{0.15(4)} \approx 18000$$

Example:

According to United Nations data, the world population at the beginning of 1990 was approximately 5.3 billion and growing at a rate of 2% per year. Assuming an exponential growth model, estimate the world population at the beginning of the year 2015.

$P = 5.3e^{0.02(25)}$

(solution: 8.7 billion)

$1990 \Rightarrow t = 0$

$2015 \Rightarrow t = 25$

$P = 8.7$

More Examples...

1. Assume that the population of the U.S. increases at a rate proportional to the population, and that the population was 150 million in 1950 and 200 million in 1970.
   a.) Estimate the population for the year 2000.

   b.) Approximately when will the population be one billion?

$$P = 150e^{Kt}$$

$$200 = 150e^{K(20)}$$

$$\frac{4}{3} = e^{20K}$$

$$\ln\left(\frac{4}{3}\right) = \ln e^{20K}$$

$$\ln\left(\frac{4}{3}\right) = 20K$$

$$K = \frac{\ln\left(\frac{4}{3}\right)}{20}$$

$$P = 150e^{\frac{\ln\left(\frac{4}{3}\right)}{20}t}$$

(a) $\underline{t = 50}$

$$P = 150e^{\frac{\ln(4/3)}{20}(50)}$$

$$\underline{P = 300\text{ million}}$$

(b) $10^6 \rightarrow$ million

$10^9 \rightarrow$ Billion

1000 million = Billion

$$1000 = 150e^{\frac{\ln\left(\frac{4}{3}\right)}{20}t}$$

$$\frac{1000}{150} = e^{\frac{\ln\left(\frac{4}{3}\right)}{20}t}$$

$$\ln\left(\frac{100}{15}\right) = \frac{\ln\left(\frac{4}{3}\right)}{20}t$$

$$\frac{20\ln\left(\frac{100}{15}\right)}{\ln\left(\frac{4}{3}\right)} = t$$

$$t = 132 \text{ years}$$

In 2082

2. One hundred fruit flies are placed in a breeding container that can support a population of at most 5000 fruit flies. If the population grows exponentially at the rate of 2% per day, how long will it take for the container to reach capacity?

$$N = Ae^{kt}$$

$$5000 = 100e^{0.02t}$$

$$\ln 50 = 0.02t$$

$$\frac{\ln 50}{0.02} = t$$

$$t \doteq \underline{196 \text{ days}}$$

3. A radioactive substance decays at a rate proportional to the amount present. Assuming the "half life" is 5 years, how ~~only~~ long will it be until only 1% of the original substance remains?

$$M = Ae^{Kt}$$

After 5 years...

$$\frac{1}{2}\cancel{A} = \cancel{A}e^{K(5)}$$

$$\ln\left(\frac{1}{2}\right) = 5K$$

$$K = \frac{\ln\left(\frac{1}{2}\right)}{5}$$

$$M = Ae^{\frac{\ln\left(\frac{1}{2}\right)}{5}t}$$

$$0.01\cancel{A} = \cancel{A}e^{\frac{\ln\left(\frac{1}{2}\right)}{5}t}$$

$$\ln(0.01) = \frac{\ln\left(\frac{1}{2}\right)}{5}t$$

$$\frac{5\ln(0.01)}{\ln\left(\frac{1}{2}\right)} = t$$

$t = 33$ years

4. A certain substance decomposes at a rate proportional to its weight. If 9 grams ofthe substance are initially present and 1 gram decomposes in the first hour, how long will it take before 5 grams have decomposed?

$m = Ae^{kt}$

$m = 9e^{kt}$

$8 = 9e^{k(1)}$

$\frac{8}{9} = e^{k}$

$\ln\left(\frac{8}{9}\right) = k$

4g Remains?

$4 = 9e^{\ln\left(\frac{8}{9}\right)t}$

$\frac{4}{9} = e^{\ln\left(\frac{8}{9}\right)t}$

$\ln\left(\frac{4}{9}\right) = \ln\left(\frac{8}{9}\right)t$

$t = \frac{\ln\left(\frac{4}{9}\right)}{\ln\left(\frac{8}{9}\right)} \doteq 6.88\,h$

Newton's Law of Cooling

The rate at which a hot body cools to the temperature of its surroundings is proportional to the temperature difference between the body and its surroundings.

ie. A cup of tea at 60 degrees above its surroundings cools degree by degree twice as rapidly as when it is 30 degrees above its surroundings.

****Note: This law also applies to warming***

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

Temperature of object

$$\frac{dT}{dt} = k(T - T_s)$$

Surrounding Temperature

where k is a constant. This could be solved as a separable differential equation, but an easier method would be to change variables...

Let $y(t) = T(t) - T_s$

$$\frac{dy}{dt} = ky$$

$$m = Ae^{kt}$$

$$y(t) = T(t) - T_s$$

$$\frac{dy}{dt}$$

$$y'(t) = T'(t) - 0$$

$$\frac{dT}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{dT}{dt}$$

This means that our initial differential equation can be expressed as...

$$\frac{dy}{dt} = ky \qquad \frac{dT}{dt} = k(T(t) - T_s)$$

Note: If the object is cooling k must be negative because the temperature is decreasing as time passes.

$$y = Ae^{kt}$$

$$T(t) - T_s = (T_0 - T_s)e^{kt}$$

$$\boxed{T(t) = T_s + (T_0 - T_s)e^{kt}}$$

Newton's Law of Cooling

Example:

A cup of water with a temperature of 95 °C is placed in a room with a constant temperature of 21 °C.

(a) Assuming Newton's Law of Cooling applies, set up and solve an initial-value problem whose solution is the temperature of the water t minutes after it is placed in the room.

(b) How many minutes will it take for the water to reach a temperature of 51 °C if it cools to 85 °C in 1 minute?

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

$$85 = 21 + (95-21)e^{k(1)}$$

$$\frac{64}{74} = e^{k}$$

$$k = \ln\left(\frac{64}{74}\right)$$

$$\therefore\ 51 = 21 + 74e^{\ln\left(\frac{64}{74}\right)t}$$

$$\frac{30}{74} = e^{\ln\left(\frac{64}{74}\right)t}$$

$$\ln\left(\frac{30}{74}\right) = \ln\left(\frac{64}{74}\right)t$$

$$t = \frac{\ln\left(\frac{30}{74}\right)}{\ln\left(\frac{64}{74}\right)}$$

$t = 6.2$ minutes

Example:

In a steel mill, rod steel at 900°C is cooled by forced air at a temperature of 20°C. The temperature of the steel after one second is 400°C. When will the steel reach a temperature of 40°C? **(solution: 4.5 seconds)**

$$400 = 20 + 880e^{K(1)}$$

$$\frac{38}{88} = e^{K}$$

$$K = \ln\left(\frac{38}{88}\right) \Longrightarrow 40 = 20 + 880e^{\ln\left(\frac{38}{88}\right)t}$$

$$\frac{20}{880} = e^{\ln\left(\frac{38}{88}\right)t}$$

$$\ln\left(\frac{2}{88}\right) = \ln\left(\frac{38}{88}\right)t$$

$$t = \frac{\ln\left(\frac{2}{88}\right)}{\ln\left(\frac{38}{88}\right)}$$

$$t = 4.5 \text{ sec}$$

Samples from the University of Texas...

1. An object is heated to 100°C (degrees Celsius) and is then allowed to cool in a room whose air temperature is 30°C.
   * (a) If the temperature of the object is 80°C after 5 minutes, when will its temperature be 50°?
   (b) Determine the elapsed time before the temperature of the object is 35°C.
2. A pizza baked at 450°F is removed from the oven at 5:00pm into a room that is a constant 70°F. Afetr 5 minutes, the pizza is at 300°F.
   * (a) At what time can you begin eating the pizza if you want its temperature to be 135°F?
   (b) Determine the time that needs to elapse before the pizza is 160°F.

1. $80 = 30 + 70e^{k(5)}$

$\frac{5}{7} = e^{5k}$

$\ln\left(\frac{5}{7}\right) = 5k \implies 50 = 30 + 70e^{\frac{\ln\left(\frac{5}{7}\right)}{5}t}$

$k = \frac{\ln\left(\frac{5}{7}\right)}{5}$ $\qquad \ln\left(\frac{2}{7}\right) = \frac{\ln\left(\frac{5}{7}\right)}{5}t$

$\frac{5\ln\left(\frac{2}{7}\right)}{\ln\left(\frac{5}{7}\right)} = t$

$t = 18.6$ minutes

2. a) $300 = 70 + 380e^{k(5)}$

$\frac{23}{38} = e^{5k}$

$\frac{\ln\left(\frac{23}{38}\right)}{5} = k \implies 135 = 70 + 380e^{\frac{\ln\left(\frac{23}{38}\right)}{5}t}$

$\frac{5\ln\left(\frac{65}{380}\right)}{\ln\left(\frac{23}{38}\right)} = t$

$t = \underline{17.6\text{ minutes}}$

$\therefore$ @ $\underline{5{:}18\text{ PM}}$

Crime Scene

A detective is called to the scene of a crime where a dead body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be 80° F. The detective checks the programmable thermostat and finds that the room has been kept at a constant 68° F for the past 3 days.

After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be 78.5° F. This last temperature reading was taken exactly one hour after the first one. The next day the detective is asked by another investigator, **"What time did our victim die?"** Assuming that the victim's body temperature was normal (98.6° F) prior to death, what is her answer to this question? Newton's Law of Cooling can be used to determine a victim's time of death.

$78.5 = 68 + 12e^{k(1)}$ $\underline{t=0} \Rightarrow 10:23$

$\frac{10.5}{12} = e^{k}$

$\underline{\ln\left(\frac{10.5}{12}\right) = k}$

$98.6 = 68 + 12e^{\ln\left(\frac{10.5}{12}\right)t}$

$\ln\left(\frac{30.6}{12}\right) = \ln\left(\frac{10.5}{12}\right)t$

$0.01\,hr \cdot \frac{60\,mi}{1\,h}$

$t = -7.01$ hours

7 hours Prior to 10:23 PM

$\therefore$ Time of death: $\underline{3:23}$

Attachments

Review - Practice Test for Sinusoidal Functions.doc

Review - Trigonometric Functions(3)(4).doc