Warm Up

Peter and Mary have purchased a home in an affluent neighbourhood for $225 000. The real estate agent informs them that homes in this area have generally appreciated by 10% every 5 years. Based on this, how much should they be able to sell their home for in 12 years?

(yrs) t	0	5
V	225 000	

$\times 1.1$

$$V = 225\,000(1.1)^{\frac{t}{5}}$$

$$V = 225\,000(1.1)^{12/5}$$

$$V = \$282\,829.67$$

Find the amount of money you will have after 10 years if \$15,000 is invested in accounts paying 6% /a interest compounded:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

a. Annually

$$A = 15000(1 + 0.06)^{10}$$
$$= \$26\,862.72$$

b. Quarterly

$$A = 15000\left(1 + \frac{0.06}{4}\right)^{40}$$
$$A = \$27\,210.28$$

t	0	0.25
A	15000	

$$\times \left(1 + \frac{0.06}{4}\right)$$

c. Monthly

d. Daily

$$A = 15000\left(1 + \frac{0.06}{12}\right)^{120}$$
$$= \$27\,290.95$$

$$A = 15000\left(1 + \frac{0.06}{365}\right)^{3650}$$
$$A = \$27\,330.43$$

When doctors prescribe medicine, they must consider how much the drug's effectiveness will decrease as time passes. If each hour a drug is only 95% as effective as the previous hour, at some point the patient will not be receiving enough medicine and must be given another dose. If the initial dose was 250 mg and the drug was administered 3 hours ago, how long will it take for the initial dose to reach a dangerously low level of 52 mg?

(approx. 30.61 hrs)

t	0	1
m (mg)	250	

$\times 0.95$

$M = 250(0.95)^t$

$52 = 250(0.95)^t$

$\frac{52}{250} = (0.95)^t$

$0.208 = 0.95^t$

$8 = 2^t$

$t = 3$

$0.95^{10} = 0.599$

$0.95^{20} = 0.358$

$0.95^{30} = 0.214$

$0.95^{31} = 0.204$

Check-Up...

Your baby brother has an ear infection. The doctor said there are probably 50,000,000 bacteria in his left ear. The penicillin the doctor prescribed will kill 7% of the bacteria every hour. How many bacteria will be in your brother's ear in 5 days?

t (hrs)	0	1
N	50000000	

$\times 0.93$

5 days = 120 hours

$N = 50\,000\,000(0.93)^{\frac{t}{1}}$

$N = 50\,000\,000(0.93)^{120}$

$N = 8259$ Bacteria

The original weight of a certain radioactive substance is 100 grams. After 2 hours it decays to 12.5 grams. Find the half-life of this substance.

t	0	2
m	100	12.5

$r = \frac{12.5}{100}$

$r = 0.125$

$\frac{125}{1000} = \frac{1}{8}$

$M = 100(0.125)^{\frac{t}{2}}$

$50 = 100(0.125)^{t/2}$

$\frac{1}{2} = 0.125^{t/2}$

$\frac{1}{2} = \left(\frac{1}{8}\right)^{t/2}$

$\frac{1}{2}^{1} = \left(\frac{1}{2}\right)^{\frac{3t}{2}}$

$1 = \frac{3t}{2}$

$\frac{2}{3} = \frac{3t}{3}$

$\frac{2}{3} = t$

$t = \frac{2}{3}$ hour $\Rightarrow$ 40 minutes

t	0	x
m	100	50

$\times \frac{1}{2}$

$M = 100\left(\frac{1}{2}\right)^{\frac{t}{x}}$

Glenn and Arlene plan to invest money for their newborn grandson so that he has \$20 000 available for his education on his 18th birthday. Assuming a growth rate of 7% per year, compounded semi-annually, how much will Glenn and Arlene need to invest today?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$20000 = P\left(1 + \frac{0.07}{2}\right)^{2(18)}$$

t	0	0.5	
V	P		

$\times 1.035$

$$V = P(1.035)^{2t}$$

$$20000 = P(1.035)^{2(18)}$$

$$P = \$\underline{5796.65}$$

Nickel-65 (Ni-65) has a half-life of 2.5 h.

How long will it take for a sample

of Ni-65 to decay to 1/1024

of its original mass?

t	0	2.5 h
m	x	$\frac{1}{2}x$

$\times \frac{1}{2}$

$1024 = 2^?$

$\log_2 1024 = ?$

$$m = x\left(\frac{1}{2}\right)^{\frac{t}{2.5}}$$

$$\frac{\frac{1}{1024}\cancel{x}}{\cancel{x}} = \frac{\cancel{x}\left(\frac{1}{2}\right)^{\frac{t}{2.5}}}{\cancel{x}}$$

$$\left(\frac{1}{2}\right)^{10} = \left(\frac{1}{2}\right)^{t/2.5}$$

$$10 = \frac{t}{2.5}$$

$t = 25$ hours

Logarithmic Functions

Did You Know?

Logarithms were developed independently by John Napier (1550–1617), from Scotland, and Jobst Bürgi (1552–1632), from Switzerland. Since Napier published his work first, he is given the credit. Napier was also the first to use the decimal point in its modern context.

Logarithms were developed before exponents were used. It was not until the end of the seventeenth century that mathematicians recognized that logarithms are exponents.

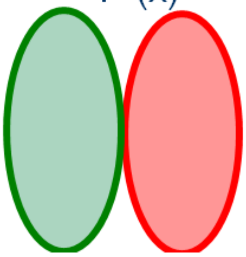
Recall some knowledge about inverse functions...

•Inverse functions is the set of ordered pair obtained by interchanging the x and y values.

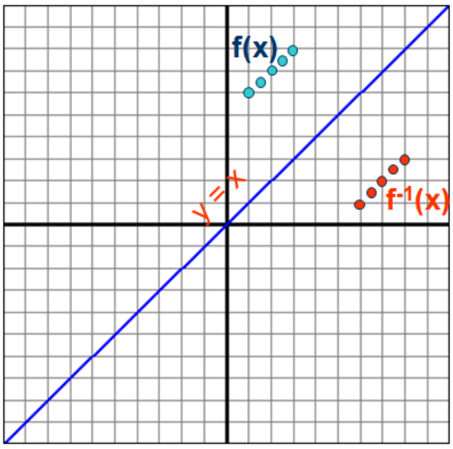
$f(x)$

x	y
2	12
3	13
4	14
5	15
6	16

$f^{-1}(x)$



•Inverse functions can be created graphically by a reflection on the y = x axis.



Word association game....

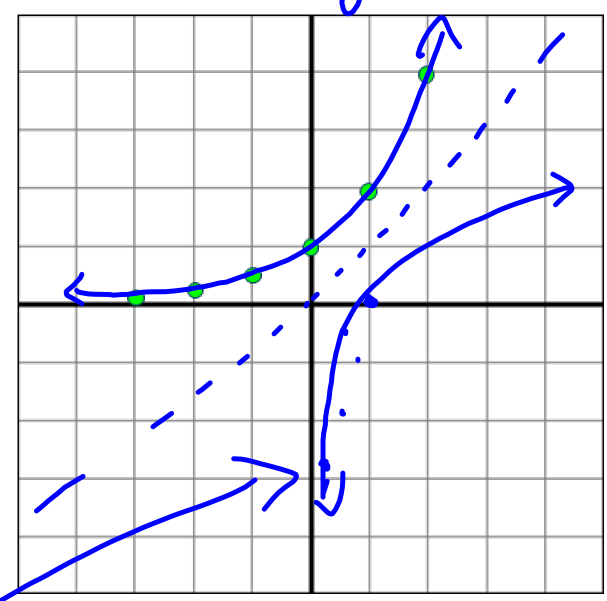
Addition ...	inverse process	...??
Multiplication ...	inverse process	...??
Squaring ...	inverse process	...??
Sine ...	inverse process	...??
Exponential ...	inverse process	...??

•Let us graph the exponential function $y = 2^x$

•Table of values:

x	y
-3	0.125
-2	0.25
-1	0.5
0	1
1	2
2	4

$y = 2^x$



$\sin \boxed{70^\circ}$

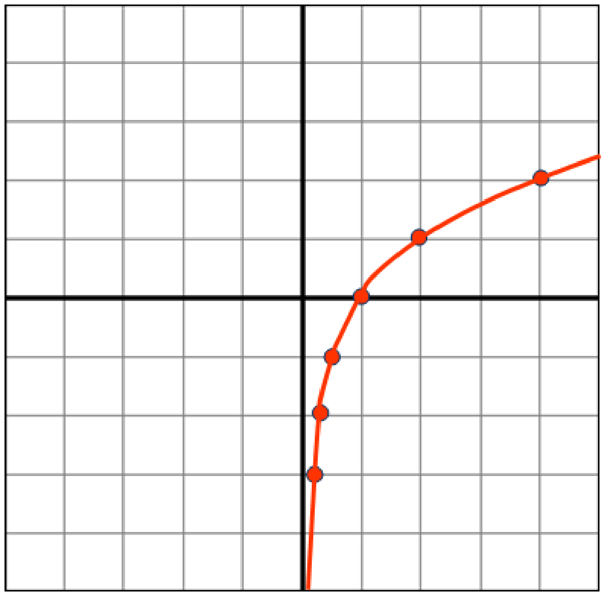
$\log_2 \boxed{x}$

$y = \log_2 \boxed{x}$ Argument

Base

- Let us find the inverse the exponential function $y = 2^x$
- Table of values:

x	y
0.125	-3
0.25	-2
0.5	-1
1	0
2	1
4	2



- Next, you will find the inverse of an exponential algebraically
- Write the process in your notes

$y = a^x$

Interchange $x \rightarrow y$

$x = a^y$

- We write these functions as:
  - $x = a^y \longleftrightarrow y = \log_a x$

log

base

$x = a^y$

exponent

$y = \log_a x$

exponent base

IMPORTANT NOTATION!!!

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

Exponential Form		Logarithmic Form	
$x = a^y$	is written as...	$y = \log_a x$	← read as... "log of x to the base a"
Say, "the base a to the exponent y is x."		Say, "y is the exponent to which you raise the base a to get the answer x."	$a > 0$ and $a \neq 1$

$\log_2 2 7$ $\log_8 37$

Example 1) Write the following into logarithmic form:

a) $3^3 = 27$ → $\log_3 27=3$

b) $4^5 = 256$ → $\log_4 256=5$

c) $2^7 = 128$ → $\log_2 128=7$

d) $(1/3)^x=27$ → $\log_{1/3} 27=x$

Example 2) Write the following into exponential form:

a) $\log_2 64=6$ → $2^6 = 64$

b) $\log_{25} 5=1/2$ → $25^{1/2} = 5$

c) $\log_8 1=0$ → $8^0 = 1$

d) $\log_{1/3} \overset{1}{9}=2$ → $(1/3)^2 = 1/9$

$\log_5 37 = x$

"5 Raised to what exponent will result in 37?"

Example 3) Find the value of x for each example:

a) $\log_{1/3}27 = x$

$\left(\frac{1}{3}\right)^{-3}$

$\left(\frac{1}{3}\right)^{x} = 27$ $\left(\frac{3}{1}\right)^{3}$

27

$(3^{-1})^{x} = 3^{3}$

$3^{-x} = 3^{3}$

$x = -3$

b) $\log_5 x = 3$

$5^3 = x$

$125 = x$

c) $\log_x(1/9) = 2$

$\sqrt{x^2} = \sqrt{\frac{1}{9}}$

$x = \frac{1}{3}$

d) $\log_3 x = 0$

$3^0 = x$

$1 = x$

What is $\log_{\square} 1000$?

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

So now after reading the above...What is log 1000 ?

$\log 17 \Rightarrow 10^{?} = 17$

$\log 17 = 1.23$

10^x

$\boxed{\log}$ ← Common Log (Base 10)

e^x

$\boxed{\ln}$ ← Natural Log (Base "e")

Euler's Number

Review...

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

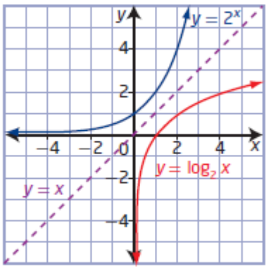
- a logarithm with base 10

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	Logarithmic Form
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.



- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.
- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in R\}$
  - the range is $\{y \mid y \in R\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

  $\log_{10} x = \log x$

What is the value of any logarithm with an argument of 1? Why?

3 Properties

General Properties of Logarithms:

If $a > 0$ and $a \neq 1$, then...

(i) $\log_a 1 = 0$

(ii) $\log_a a^x = x$

(iii) $a^{\log_a x} = x$

$$a^{\log_a x} = M$$

$$\log_a M = \log_a x$$

$$\therefore M = x$$

$$7^x = 7^4$$

$$7^{\log_7 38} = 38$$

Check-Up Time...

Evaluate each of the following:

1. $\log_2 8\sqrt{32} = x$

$2^x = 8\sqrt{32}$

$2^x = 2^3 (2^5)^{1/2}$

$2^x = 2^3 \cdot (2)^{5/2}$

$2^x = 2^{11/2}$

$x = \frac{11}{2}$

$\log_2 2^{\frac{11}{2}}$

$= \frac{11}{2}$

2. $-\frac{2}{3} = \log_x 81$

$\left(x^{-2/3}\right)^{-3/2} = (81)^{-\frac{3}{2}}$

$x^1 = \frac{1}{(\sqrt{81})^3} = \frac{1}{729}$

3. $\log_5 \frac{1}{125}$

$\log_5 5^{-3}$

$= -3$

$\log_b b^M$

$= \underline{\underline{M}}$

4. $\log_{\sqrt{6}} 36 = x$

$(\sqrt{6})^x = 36$

$6^{\frac{1}{2}x} = 6^2$

$\frac{1}{2}x = 2$

$\underline{x = 4}$

Laws of Logarithms

Did You Know?

The world's first hand-held scientific calculator was the Hewlett-Packard HP-35, so called because it had 35 keys. Introduced in 1972, it retailed for approximately U.S. $395. Market research at the time warned that the demand for a pocket-sized calculator was too small. Hewlett-Packard estimated that they needed to sell 10 000 calculators in the first year to break even. They ended up selling 10 times that. By the time it was discontinued in 1975, sales of the HP-35 exceeded 300 000.

Laws of Logarithms: If ***a > 0, M > 0, N > 0 and n ε R then…***

1) Product Law → the logarithm of a product is equal to the sum of the logarithms of the factors.

PROOF: Let $\log_a M = b$ and $\log_a N = c$

so $a^b = M$ $a^c = N$

then,

$$\log_a(MN) = \log_a(a^b \bullet a^c)$$
$$= \log_a(a^{b+c})$$
$$= b + c$$
$$\therefore \boxed{\log_a(MN) = \log_a M + \log_a N}$$

examples: a) $\log_{10}(6 \times 9)$

log 6 + log 9

b) $\log_2 12 + \log_2 7$

$\log_2(12 \times 7)$

$\log_2(84)$

2) Quotient Law → the logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.

PROOF: Let $\log_a M = b$ and $\log_a N = c$

so $a^b = M$ $a^c = N$

then,

$$\log_a\left(\frac{M}{N}\right) = \log_a\left(\frac{a^b}{a^c}\right)$$
$$= \log_a(a^{b-c})$$
$$= b - c$$

$$\therefore \boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N}$$

examples: a) $\log_5\left(\frac{97}{62}\right)$ b) $\log_2 15 - \log_2 3$ c) $\log_{10}\left(\frac{1}{9}\right)$

$\log_5 97 - \log_5 62$

$\log_2\left(\frac{15}{3}\right)$

$\log_2 5$

$\log 1 - \log 9$

$0 - \log 9$

$= -\log 9$

$$\frac{\log_3 x}{\log_3 7} \cancel{= \log_3 x - \log_3 7}$$

3) Law of Logarithms for Powers → the logarithm of a power of a number is equal to the exponent multiplied by the logarithm of the number

PROOF: Let $\log_a M = b$

so $a^b = M$

then,

$$\log_a M^p = \log_a (a^b)^p$$
$$= \log_a (a^{b \times p})$$
$$= b \times p$$

$$\therefore \boxed{\log_a M^p = p \times \log_a M}$$

examples: a) $\log_{10} 8^9$ b) $2\log_3 5$ c) $\log_5 \sqrt{125}$

a) $9\log 8$

b) $= \log_3 5^2$
$= \log_3 25$

c) $\frac{1}{2} \log_5 125$
$\frac{1}{2}(3)$
$= \frac{3}{2}$

$\log^3 x$

$(\log x)^3$

Not Power Law

$\sin^2 x = (\sin x)^2$

$\log x^3$

$= 3\log x$

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and

a) $\log_6 \frac{x}{y}$

b) $\log_5 \sqrt{xy}$

c) $\log_3 \frac{9}{\sqrt[3]{x^2}}$

d) $\log_7 \frac{x^5 y}{\sqrt{z}}$

(a) $\log_6 x - \log_6 y$

(b) $\log_5 (xy)^{\frac{1}{2}}$ $\qquad \log_5 (x^{\frac{1}{2}} y^{\frac{1}{2}})$

$\frac{1}{2} \log_5 (xy)$ $\qquad \log_5 x^{\frac{1}{2}} + \log_5 y^{\frac{1}{2}}$

$\frac{1}{2} (\log_5 x + \log_5 y) = \frac{1}{2} \log_5 x + \frac{1}{2} \log_5 y$

c) $\log_3 \left(\frac{9}{\sqrt[3]{x^2}} \right)$

Attachments

Review - Practice Test for Sinusoidal Functions.doc

Review - Trigonometric Functions(3)(4).doc