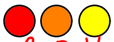
1.1 Making Conjectures: Inductive Reasoning

GOAL

Use reasoning to make predictions.

EXPLORE...

• If the first three colours in a sequence are red, orange, and yellow, what colours might be found in the rest of the sequence? Explain.

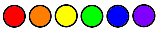


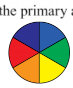
ROY G BIV

SAMPLE ANSWER

Here are three possible answers:

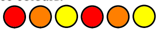
- If the colour sequence is red, orange, and yellow, the rest of the sequence may be green, blue, and purple. These colours are the primary and secondary colours seen on a colour wheel.





- If the colour sequence is red, orange, and yellow, the rest of the sequence may be green, blue, indigo, and violet. These colours are those of a rainbow.

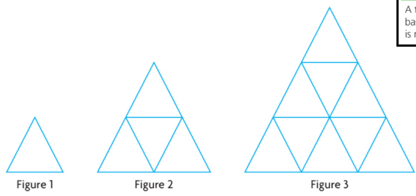
- If the colour sequence is red, orange, and yellow, the rest of the sequence may repeat these three colours.



INVESTIGATE the Math

Georgia, a fabric artist, has been patterning with equilateral triangles. Consider Georgia's **conjecture** about the following pattern.

> **conjecture**
> A testable expression that is based on available evidence but is not yet proved.



I think Figure 10 in this pattern will have 100 triangles, and all these triangles will be congruent to the triangle in Figure 1.

? How did Georgia arrive at this conjecture?

A. Organize the information about the pattern in a table.

Figure	1	2	3	4	5					
Number of Triangles	1	4								

B. With a partner, discuss what you notice about the data in the table.

C. Extend the pattern for two more figures.

D. What numeric pattern do you see in the table?

Answers

A.

B.

C.

D.

Reflecting

E. Is Georgia's conjecture reasonable? Explain.

F. How did Georgia use **inductive reasoning** to develop her conjecture?

G. Is there a different conjecture you could make based upon the pattern you see? Explain.

> **inductive reasoning**
> Drawing a general conclusion by observing patterns and identifying properties in specific examples.

Answers

E.

F.

G.

EXAMPLE 2 Using inductive reasoning to develop a conjecture about integers

Make a conjecture about the product of two odd integers. $57 \times 91 = 5187$

Jay's Solution $9 \times 9 = 81$ $7 \times 3 = 21$ $5 \times 5 = 25$ $9 \times 13 = 117$

$(+3)(+7) = (+21)$ — Odd integers can be negative or positive. I tried two positive odd integers first. The product was positive and odd.

$(-5)(-3) = (+15)$ — Next, I tried two negative odd integers. The product was again positive and odd.

$(+3)(-3) = (-9)$ — Then I tried the other possible combination: one positive odd integer and one negative odd integer. This product was negative and odd.

My conjecture is that the product of two odd integers is an odd integer. — I noticed that each pair of integers I tried resulted in an odd product.

$(-211)(-17) = (+3587)$ — I tried other integers to test my conjecture. The product was again odd.

EXAMPLE 2 Using inductive reasoning to develop a conjecture about integers

Make a conjecture about the product of two odd integers.

Your Turn

Do you find Jay's conjecture convincing? Why or why not?

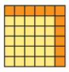
Answer

EXAMPLE 3 Using inductive reasoning to develop a conjecture about perfect squares

Make a conjecture about the difference between consecutive perfect squares. $4^2, 5^2$

Steffan's Solution: Comparing the squares geometrically $25-16=9$

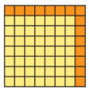
I represented the difference using unit tiles for each perfect square. First, I made a 3 × 3 square in orange and placed a yellow 2 × 2 square on top. When I subtracted the 2 × 2 square, I had 5 orange unit tiles left.



Next, I made 3 × 3 and 4 × 4 squares. When I subtracted the 3 × 3 square, I was left with 7 orange unit tiles. I decided to try greater squares.

My conjecture is that the difference between consecutive squares is always an odd number.

I saw the same pattern in all my examples: an even number of orange unit tiles bordering the yellow square, with one orange unit tile in the top right corner. So, there would always be an odd number of orange unit tiles left, since an even number plus one is always an odd number.



I tested my conjecture with the perfect squares 7 × 7 and 8 × 8. The difference was an odd number.

The example supports my conjecture.

$25-16=9$

4^2-3^2
$16-9$
7

11^2-10^2
$121-100$
21

$22^2-21^2=43$

P12 #3,4,5

EXAMPLE 3 Using inductive reasoning to develop a conjecture about perfect squares

Make a conjecture about the difference between consecutive perfect squares.

Francesca's Solution: Describing the difference numerically

$2^2 - 1^2 = 4 - 1$
$2^2 - 1^2 = 3$

I started with the smallest possible perfect square and the next greater perfect square: 1^2 and 2^2. The difference was 3.

$4^2 - 3^2 = 7$
$9^2 - 8^2 = 17$

Then I used the perfect squares 3^2 and 4^2. The difference was 7. So, I decided to try even greater squares.

My conjecture is that the difference between consecutive perfect squares is always a prime number.

I thought about what all three differences—3, 7, and 17—had in common. They were all prime numbers.

$12^2 - 11^2 = 23$

To test my conjecture, I tried the perfect squares 11^2 and 12^2. The difference was a prime number.

The example supports my conjecture.

EXAMPLE 3 Using inductive reasoning to develop a conjecture about perfect squares

Make a conjecture about the difference between consecutive perfect squares.

Your Turn

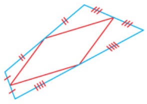
How is it possible to have two different conjectures about the same situation? Explain.

Answer

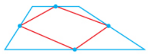
EXAMPLE 4 Using inductive reasoning to develop a conjecture about quadrilaterals

Make a conjecture about the shape that is created by joining the midpoints of adjacent sides in any quadrilateral.

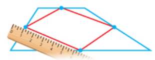
Marc's Solution: Using a protractor and ruler



I drew an irregular quadrilateral on tracing paper. I used my ruler to determine the midpoints of each side. I joined the midpoints of adjacent sides to form a new quadrilateral. This quadrilateral looked like a parallelogram.



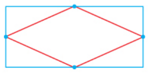
Next, I drew a trapezoid with sides that were four different lengths. I determined the midpoints of the sides. When the midpoints were joined, the new quadrilateral looked like a parallelogram.



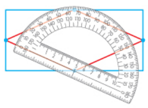
I used my ruler to confirm that the opposite sides were equal.

My conjecture is that joining the adjacent midpoints of any quadrilateral will create a parallelogram.

Each time I joined the midpoints, a parallelogram was formed.



To check my conjecture one more time, I drew a rectangle. I determined its midpoints and joined them. This quadrilateral also looked like a parallelogram.



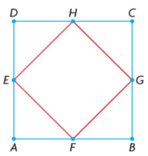
I checked the measures of the angles in the new quadrilateral. The opposite angles were equal. The new quadrilateral was a parallelogram, just like the others were.

The rectangle example supports my conjecture.

EXAMPLE 4 Using inductive reasoning to develop a conjecture about quadrilaterals

Make a conjecture about the shape that is created by joining the midpoints of adjacent sides in any quadrilateral.

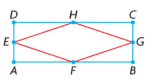
Tracey's Solution: Using dynamic geometry software



$\overline{HE} = 2.4$ cm
$\overline{EF} = 2.4$ cm
$\overline{FG} = 2.4$ cm
$\overline{GH} = 2.4$ cm

$\angle EFG = 90°$
$\angle FGH = 90°$
$\angle GHE = 90°$
$\angle HEF = 90°$

I constructed a square and the midpoints of the sides. Then I joined the adjacent midpoints. *EFGH* looked like a square. I checked its side lengths and angle measures to confirm that it was a square.



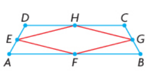
$\overline{HE} = 1.6$ cm
$\overline{EF} = 1.6$ cm
$\overline{FG} = 1.6$ cm
$\overline{GH} = 1.6$ cm

$\angle EFG = 143°$
$\angle FGH = 37°$
$\angle GHE = 143°$
$\angle HEF = 37°$

Next, I constructed a rectangle and joined the adjacent midpoints to create a new quadrilateral, *EFGH*. The side lengths and angle measures of *EFGH* showed that *EFGH* was a rhombus but not a square.

My conjecture is that the quadrilateral formed by joining the adjacent midpoints of any quadrilateral is a rhombus.

Since a square is a rhombus with right angles, both of my examples resulted in a rhombus.



$\overline{HE} = 1.6$ cm
$\overline{EF} = 1.6$ cm
$\overline{FG} = 1.6$ cm
$\overline{GH} = 1.6$ cm

$\angle EFG = 152°$
$\angle FGH = 28°$
$\angle GHE = 152°$
$\angle HEF = 28°$

To check my conjecture, I tried an isosceles trapezoid. The new quadrilateral, *EFGH*, was a rhombus.

The isosceles trapezoid example supports my conjecture.

EXAMPLE 4 Using inductive reasoning to develop a conjecture about quadrilaterals

Make a conjecture about the shape that is created by joining the midpoints of adjacent sides in any quadrilateral.

Your Turn

a) Why did the students draw different conjectures?

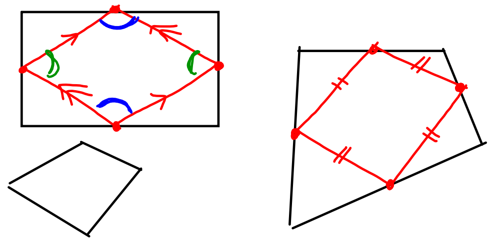
b) Do you think that both conjectures are valid? Explain.

Answers

a)

b)

rhombus



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3. Sum of Two Even Numbers

$2+4=6$

$8+8=16$

$4+6=10$

$14+20=34$

$(-2)+(4)=2$

$(-10)+(-6)=-16$

$0 \rightarrow$ "even" neither!

$(-4)+(4)=0$ not even

In Summary

Key Idea

- Inductive reasoning involves looking at specific examples. By observing patterns and identifying properties in these examples, you may be able to make a general conclusion, which you can state as a conjecture.

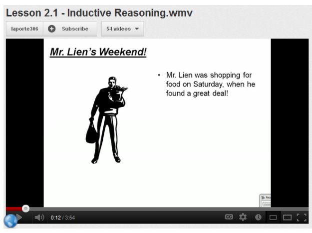
Need to Know

- A conjecture is based on evidence you have gathered.
- More support for a conjecture strengthens the conjecture, but does not prove it.

HW...

p. 12: #1 - 3; #6 - 11; 13; 15; 16

REVIEW...



1.3 Using Reasoning to Find a Counterexample to a Conjecture

GOAL

Invalidate a conjecture by finding a contradiction.

To restate what you have read so far, a conjecture is a mathematical statement that has been proposed as a true statement, but not yet proven or disproved.

Once a conjecture is proven, it is a mathematical **fact**.

One method to test a conjecture is to attempt to **disprove** it by using a **counterexample**.

13 / 1

For example:

Conjecture: All prime numbers are odd.
Counterexample: But 2 is a prime number.

2 / 1 ~~21~~

The counterexample disproves the conjecture, hence we can conclude that not all prime numbers are odd.

EXAMPLE #2:

$(0.5)^2 = 0.25$

Conjecture:

For all real numbers x, the expressions x^2 is greater than or equal to x

Conjecture: For all real numbers x, the expressions x^2 is greater than or equal to x

COUNTEREXAMPLE???

$(0.9)^2 = 0.81$

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EXAMPLE 3 Using reasoning to find a counterexample to a conjecture

Matt found an interesting numeric pattern:

$1 \cdot 8 + 1 = \mathbf{9}$
$12 \cdot 8 + 2 = \mathbf{98}$
$123 \cdot 8 + 3 = \mathbf{987}$
$1234 \cdot 8 + 4 = \mathbf{9876}$

$12345 \cdot 8 + 5 = 98765$

Matt thinks that this pattern will continue.
Search for a counterexample to Matt's conjecture.

Kublu's Solution

$\mathbf{1} \cdot 8 + \mathbf{1} = 9$
$\mathbf{12} \cdot 8 + \mathbf{2} = 98$
$\mathbf{123} \cdot 8 + \mathbf{3} = 987$
$\mathbf{1234} \cdot 8 + \mathbf{4} = 9876$

The pattern seemed to be related to the first factor (the factor that wasn't 8), the number that was added, and the product.

	A	B
1	1 · 8 + 1	9
2	12 · 8 + 2	98
3	123 · 8 + 3	987
4	1234 · 8 + 4	9876
5	12345 · 8 + 5	98765
6	123456 · 8 + 6	987654
7	1234567 · 8 + 7	9876543
8	12345678 · 8 + 8	98765432
9	123456789 · 8 + 9	987654321

I used a spreadsheet to see if the pattern continued.

The spreadsheet showed that it did.

$123456789\mathbf{10} \cdot 8 + \mathbf{10} = 98\,765\,431\,290$
$12345678\mathbf{90} \cdot 8 + \mathbf{10} = 9\,876\,543\,130$
$123456789\mathbf{10} \cdot 8 + \mathbf{0} = 98\,765\,431\,280$
$123456789\mathbf{0} \cdot 8 + \mathbf{0} = 9\,876\,543\,120$

When I came to the tenth step in the sequence, I had to decide whether to use 10 or 0 in the first factor and as the number to add. I decided to check each way that 10 and 0 could be represented.

The pattern holds true until 9 of the 10 digits are included. At the tenth step in the sequence, a counterexample is found.

Since the pattern did not continue, Matt's conjecture is invalid.

Revised conjecture: When the value of the addend is 1 to 9, the pattern will continue.

I decided to revise Matt's conjecture by limiting it.

Your Turn

If Kublu had not found a counterexample at the tenth step, should she have continued looking? When would it be reasonable to stop gathering evidence if all the examples supported the conjecture? Justify your decision.

Answer

Mathematical HISTORY???

Goldbach's Conjecture

One famous example of an unproven conjecture has remained undecided for nearly 300 years.

In the early 1700's, Christian Goldbach, a Prussian mathematician, noticed that many even numbers greater than 2 can be written as the sum of two primes. Expanding on examples like these, Goldbach wrote the following conjecture:

4 = 2 + 2	10 = 3 + 7	16 = 3 + 13
6 = 3 + 3	12 = 5 + 7	18 = 5 + 13
8 = 3 + 5	14 = 3 + 11	20 = 3 + 17

Conjecture: Every even number greater than 2 can be written as the sum of two primes.

To this day, no one has proven **Goldbach's Conjecture** or found a counterexample to show that it is false. It is still unknown whether this conjecture is true or false. It is known, however, that all even numbers up to 4×10^{18} confirm Goldbach's Conjecture.

Christian Goldbach (1690 – 1764) was a German mathematician famous for his eponymous Conjecture. Goldbach's Conjecture is one of the most infamous problems in mathematics, and states that every even integer number greater than 2 can be expressed as the sum of two prime numbers. For example, 4=2+2, 6=3+3, and 8=3+5. While there have not been any counter-examples found up through 4×10^{18}(as of 2012), the conjecture has not yet been formally proven.

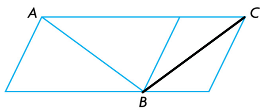
1.2 Exploring the Validity of Conjectures

GOAL

Determine whether a conjecture is valid.

EXPLORE the Math p. 16

Your brain can be deceived.

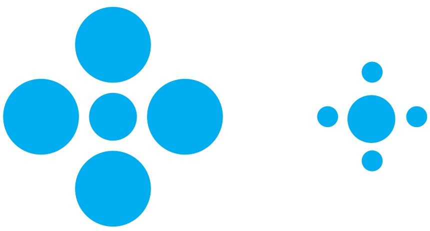


Make a conjecture about diagonal *AB* and diagonal *BC*.

? How can you check the validity of your conjecture?

EXPLORE the Math

Your brain can be deceived.

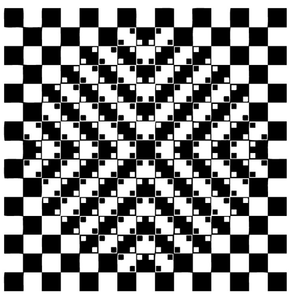


Make a conjecture about the circles in the centre.

? How can you check the validity of your conjecture?

EXPLORE the Math

Your brain can be deceived.

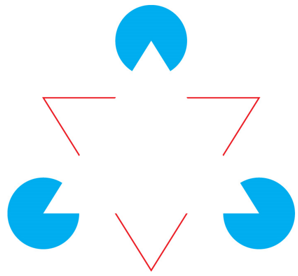


Make a conjecture about the lines.

? How can you check the validity of your conjecture?

EXPLORE the Math

Your brain can be deceived.



Make a conjecture about the number of triangles.

How can you check the validity of your conjecture?

Reflecting

A. Describe the steps you took to verify your conjectures.

B. After collecting evidence, did you decide to revise either of your conjectures? Explain.

C. Can you be certain that the evidence you collect leads to a correct conjecture? Explain.

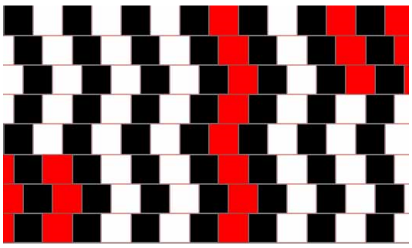
Answers

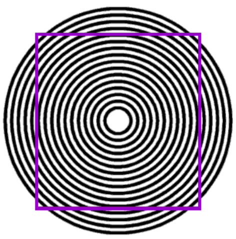
A.

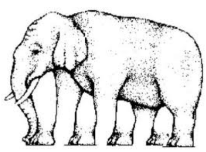
B.

C.

Some other optical illusions...







M. C. Escher...

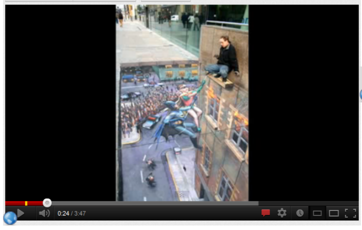
http://www.mcescher.com/

Three Dragons

three_dragons.wmv

3D Chalk Art.. Julian Beever

extreme 3D sidewalk chalk drawings.



"Julian Beever is an English, Belgium-based chalk artist who has been creating trompe-l'œil chalk drawings on pavement surfaces since the mid-1990s. His works are created using a projection called anamorphosis, and create the illusion of three dimensions when viewed from the correct angle."

44 Amazing Julian Beever's 3D Pavement Drawings

1.2 - Validity of Conjectures?

In Summary page 17

Key Idea

- Some conjectures initially seem to be valid, but are shown not to be valid after more evidence is gathered.

Need to Know

- The best we can say about a conjecture reached through inductive reasoning is that there is evidence either to support or deny it.
- A conjecture may be revised, based on new evidence.

1.3 - Counterexamples

In Summary page 22

Key Ideas

- Once you have found a counterexample to a conjecture, you have disproved the conjecture. This means that the conjecture is invalid.
- You may be able to use a counterexample to help you revise a conjecture.

Need to Know

- A single counterexample is enough to disprove a conjecture.
- Even if you cannot find a counterexample, you cannot be certain that there is not one. Any supporting evidence you develop while searching for a counterexample, however, does increase the likelihood that the conjecture is true.

HOMEWORK...

p. 17: #1 & 2
p. 22: #1, 3, 5, 8, 12, 17

1.4 Proving Conjectures: Deductive Reasoning

GOAL

Prove mathematical statements using a logical argument.

Every day, you use **deductive thinking** to **deduce** new information.

In this course, you will use this method to deduce the properties of geometric figures and many geometric relationships. For example:

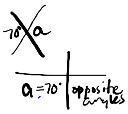
Step A General Statement	And	Step B Particular Statement	Thus	Step C Conclusion
During a game, five players are used on a basketball team.	And	UNB is playing basketball.	Thus	UNB uses five players on the basketball team.
All isosceles triangles have two equal sides.	And	Triangle ABC is isosceles.	Thus	Two sides of Triangle ABC are equal.

In Step A, based on your earlier knowledge (your experience, what you have learned in life) you accept certain general statements to be true. In Step B you are confronted with a particular case that is related to a general statement. Lastly, in Step C, you deduce a conclusion based upon Step A and B.

KEY TERMS...

proof
A mathematical argument showing that a statement is valid in all cases, or that no counterexample exists.

deductive reasoning
Drawing a specific conclusion through logical reasoning by starting with general assumptions that are known to be valid.



generalization
A principle, statement, or idea that has general application.

two-column proof
A presentation of a logical argument involving deductive reasoning in which the statements of the argument are written in one column and the justifications for the statements are written in the other column.

We can often use the **transitive property** in deductive reasoning. According to this property, **if two things are equal to the same thing, then they are equal to one another**. We can express this property mathematically:

If $a = b$ and $b = c$, then $a = c$.

b = a
b = c
∴ a = c

y = 2x + 1
y = -1/3x + 8
2x + 1 = -1/3x + 8

↖ triangle with dots
"therefore"

Mathematical Proofs

A **proof** is a convincing argument that something is true.

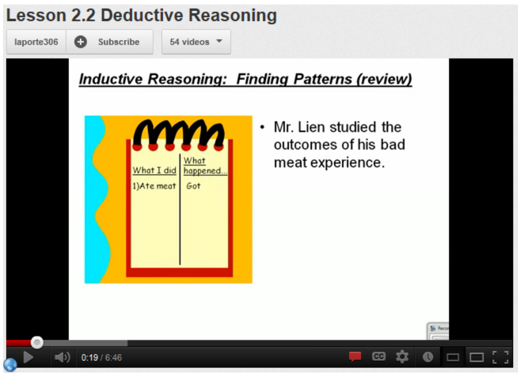
In mathematics, a proof starts with things that are agreed upon, called **postulates** or **axioms**, and then uses logic to reach a conclusion. Conclusions are often reached in geometry by observing data and looking for patterns. As you learned earlier, this type of reasoning is called **inductive reasoning** and the conclusion reached by inductive reasoning is called a **conjecture**.

A proof in geometry consists of a sequence of statements—each supported by a reason—that starts with a given set of premises and leads to a valid conclusion. This type of reasoning is called **deductive reasoning**. Each statement in a proof follows from one or more of the previous statements.

A **reason** (a fact) for a statement can come from the set of given premises or from one of the four types of other premises:

- Definitions
- Postulates
- Properties of algebra, equality or congruence
- Previously proven theorems

Once a conjecture is proved, it is called a **theorem**. As a theorem, it becomes a premise for geometric arguments you can use to prove other conjectures.



LEARN ABOUT the Math

Jon discovered a pattern when adding integers:

3×5

$$1 + 2 + 3 + 4 + 5 = 15$$
$$(-15) + (-14) + (-13) + (-12) + (-11) = -65$$
$$(-3) + (-2) + (-1) + 0 + 1 = -5$$

He claims that whenever you add five consecutive integers, the sum is always 5 times the median of the numbers.

How can you prove that Jon's conjecture is true for all integers?

p. 27

EXAMPLE 1 Connecting conjectures with reasoning

Prove that Jon's conjecture is true for all integers.

Pat's Solution

$$5(3) = 15$$
$$5(-13) = -65$$
$$5(-1) = -5$$

The median is the middle number in a set of integers when the integers are arranged in consecutive order. I observed that Jon's conjecture was true in each of his examples.

$$210 + 211 + 212 + 213 + 214 = 1060$$
$$5(212) = 1060$$

I tried a sample with greater integers, and the conjecture still worked.

Let x represent any integer.
Let S represent the sum of five consecutive integers.
$S = (x - 2) + (x - 1) + x + (x + 1) + (x + 2)$

I decided to start my proof by representing the sum of five consecutive integers. I chose x as the median and then wrote a generalization for the sum.

proof
A mathematical argument showing that a statement is valid in all cases, or that no counterexample exists.

generalization
A principle, statement, or idea that has general application.

$S = (x + x + x + x + x) + (-2 + (-1) + 0 + 1 + 2)$
$S = 5x + 0$

I simplified by gathering like terms.

$S = 5x$
Jon's conjecture is true for all integers.

Since x represents the median of five consecutive integers, $5x$ will always represent the sum.

$x + x+1 + x+2 + x+3 + x+4 = 5x+10$

$5(x+2)$

$5x+10$

① x

② $4x$

③ $\dfrac{\frac{4x+10}{2} - 5}{2}$

$\dfrac{2x+5-5}{2}$ $\quad \dfrac{2x}{2} = x$

1st 2nd 3rd 4th 5th

$x + (x+1) + (x+2) + (x+3) + (x+4) = 5x+10$

median

$5(x+2)$

$5x+10$

Reflecting

A. What type of reasoning did Jon use to make his conjecture?

B. Pat used **deductive reasoning** to prove Jon's conjecture. How does this differ from the type of reasoning that Jon used?

deductive reasoning
Drawing a specific conclusion through logical reasoning by starting with general assumptions that are known to be valid.

Answers

A.

B.

APPLY the Math p. 28

EXAMPLE 2 Using deductive reasoning to generalize a conjecture

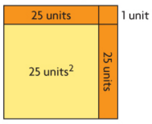
In Lesson 1.3, page 19, Luke found more support for Steffan's conjecture from Lesson 1.1, page 9—that the difference between consecutive perfect squares is always an odd number.

Determine the general case to prove Steffan's conjecture.

Back to previous lesson...

Gord's Solution

The difference between consecutive perfect squares is always an odd number. — Steffan's conjecture has worked for consecutive perfect squares with sides of 1 to 7 units.



$26^2 - 25^2 = 2(25) + 1$
$26^2 - 25^2 = 51$

I tried a sample using even greater squares: 26^2 and 25^2. The difference is the two sets of 25 unit tiles, plus a single unit tile.

Let x be any natural number.
Let D be the difference between consecutive perfect squares.
$D = (x + 1)^2 - x^2$

Since the conjecture has been supported with specific examples, I decided to express the conjecture as a general statement. I chose x to be the length of the smaller square's sides. The larger square's sides would then be $x + 1$.

$D = x^2 + x + x + 1 - x^2$
$D = x^2 + 2x + 1 - x^2$
$D = 2x + 1$

I expanded and simplified my expression. Since x represents any natural number, $2x$ is an even number, and $2x + 1$ is an odd number.

Steffan's conjecture, that the difference of consecutive perfect squares is always an odd number, has been proved for all natural numbers.

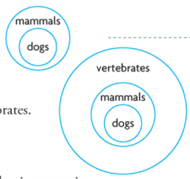
p. 29

EXAMPLE 3 Using deductive reasoning to make a valid conclusion

All dogs are mammals. All mammals are vertebrates. Shaggy is a dog.
What can be deduced about Shaggy?

Oscar's Solution

Shaggy is a dog.
All dogs are mammals.



These statements are given. I represented them using a Venn diagram.

All mammals are vertebrates.

This statement is given. I modified my diagram.

Therefore, through deductive reasoning, Shaggy is a mammal and a vertebrate.

Transitive Property...

a = b AND b = c therefore, a = c

p. 30

EXAMPLE 5 Communicating reasoning about a divisibility rule

The following rule can be used to determine whether a number is divisible by 3:

Add the digits, and determine if the sum is divisible by 3. If the sum is divisible by 3, then the original number is divisible by 3.

Use deductive reasoning to prove that the divisibility rule for 3 is valid for two-digit numbers.

17 = 8
15 = 6
144 = 9

Lee's Solution

Expanded Number Forms		
Number	Expanded Form (Words)	Expanded Form (Numbers)
9	9 ones	9(1)
27	2 tens and 7 ones	2(10) + 7(1)
729	7 hundreds and 2 tens and 9 ones	7(100) + 2(10) + 9(1)
ab	*a* tens and *b* ones	*a*(10) + *b*(1)

a b

5421 = 12

Let ab represent any two-digit number. — I let ab represent any two-digit number.

$ab = 10a + b$ — Since any number can be written in expanded form, I wrote ab in expanded form.

$ab = (9a + 1a) + b$
$ab = 9a + (a + b)$

I decomposed $10a$ into an equivalent sum. I used $9a$ because I knew that $9a$ is divisible by 3, since 3 is a factor of 9.

The number ab is divisible by 3 only when $(a + b)$ is divisible by 3.

From this equivalent expression, I concluded that ab is divisible by 3 only when both $9a$ and $(a + b)$ are divisible by 3. I knew that $9a$ is always divisible by 3, so I concluded that ab is divisible by 3 only when $(a + b)$ is divisible by 3.

The divisibility rule has been proved for two-digit numbers.

Let's do one together...

In Summary p. 31

Key Idea

- Deductive reasoning involves starting with general assumptions that are known to be true and, through logical reasoning, arriving at a specific conclusion.

Need to Know

- A conjecture has been proved only when it has been shown to be true for every possible case or example. This is accomplished by creating a proof that involves general cases.
- When you apply the principles of deductive reasoning correctly, you can be sure that the conclusion you draw is valid.
- The transitive property is often useful in deductive reasoning. It can be stated as follows: Things that are equal to the same thing are equal to each other. If $a = b$ and $b = c$, then $a = c$.
- A demonstration using an example is *not* a proof.

HOMEWORK...

p. 31: #1, 2
#4, 5
~~#7, 8~~
~~#10, 11~~
~~#15, 17~~

#8, 9, 10

WARM-UP...

1. Grab a calculator. (you won't be able to do this one in your head)
2. Key in the first three digits of your phone number (NOT the area code)
3. Multiply by 80
4. Add 1
5. Multiply by 250
6. Add the last 4 digits of your phone number
7. Add the last 4 digits of your phone number again.
8. Subtract 250
9. Divide number by 2

Do you recognize the answer?

WHY??? Prove by deduction...

1.5 Proofs That Are Not Valid

NOTE: Watch for...
- sentences that use the word ***all***
- division of zero

REMEMBER: Ask yourself does it make sense?

GOAL

Identify errors in proofs.

Logical Errors

Although deductive reasoning seems rather simple, it can go wrong in more than one way. Deductive reasoning based on incorrect premises leads to faulty conclusions. Similarly, a single error in reasoning will result in an invalid or unsupported conclusion, destroying a deductive proof.

Everyday situations are filled with examples of incorrect deductive reasoning, or **logical errors**.

Common logical errors include:

- A false assumption or generalizing
- An error in reasoning, like division by zero
- An error in calculation

Your Turn

Zack is a high school student. All high school students dislike cooking. Therefore, Zack dislikes cooking. Where is the error in the reasoning?

Communication | *Tip*

Stereotypes are generalizations based on culture, gender, religion, or race. There are always counterexamples to stereotypes, so conclusions based on stereotypes are not valid.

Answer

EXAMPLE #1...

A **fallacy** is an incorrect conclusion arrived at by apparently correct, though flawed, reasoning. Such misleading or deceptive reasoning is called **specious reasoning**.

The most common example of a mathematical fallacy is the following specious proof that 1 = 2.

Let $a = b$.
Then: $ab = a^2$
$ab-b^2 = a^2-b^2$
$b(a-b) = (a+b)(a-b)$
$b = 2b$
$1 = 2$

Solution...

The error that makes this "proof" incorrect occurs in the following step, where each side is divided by $(a-b)$. Since $a = b$ in this "proof," then $a-b = 0$, and dividing by zero is not permitted in algebra.

$b(a-b) = (a+b)(a-b)$
$b = 2b$

EXAMPLE 2 Using reasoning to determine the validity of a proof

Bev claims he can prove that $3 = 4$.

Bev's Proof

Suppose that: $a + b = c$
This statement can be written as: $4a - 3a + 4b - 3b = 4c - 3c$
After reorganizing, it becomes: $4a + 4b - 4c = 3a + 3b - 3c$
Using the distributive property, $4(a + b - c) = 3(a + b - c)$
Dividing both sides by $(a + b - c)$, $4 = 3$

Show that Bev has written an **invalid proof**.

invalid proof
A proof that contains an error in reasoning or that contains invalid assumptions.

Pru's Solution

Suppose that:
$a + b = c$ ✔

Bev's **premise** was made at the beginning of the proof. Since variables can be used to represent any numbers, this part of the proof is valid.

premise
A statement assumed to be true

$4a - 3a + 4b - 3b = 4c - 3c$ ✔

Bev substituted $4a - 3a$ for a since $4a - 3a = a$. Bev substituted $4b - 3b$ for b since $4b - 3b = b$. Bev substituted $4c - 3c$ for c since $4c - 3c = c$.

$4a + 4b - 4c = 3a + 3b - 3c$ ✔

I reorganized the equation and I came up with the same result that Bev did when he reorganized. Simplifying would take me back to the premise. This part of the proof is valid.

$4(a + b - c) = 3(a + b - c)$ ✔

Since each side of the equation has the same coefficient for all the terms, factoring both sides is a valid step.

$$\frac{4(a + b - c)}{(a + b - c)} = \frac{3(a + b - c)}{(a + b - c)}$$

This step appears to be valid, but when I looked at the divisor, I identified the flaw.

$a + b = c$
$a + b - c = c - c$
$a + b - c = 0$

When I rearranged the premise, I determined that the divisor equalled zero.

Dividing both sides of the equation by $a + b - c$ is not valid. Division by zero is undefined.

EXAMPLE 3 Using reasoning to determine the validity of a proof

Liz claims she has proved that $-5 = 5$.

Liz's Proof

I assumed that $-5 = 5$.
Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$
This means that my assumption, $-5 = 5$, must be correct.

Where is the error in Liz's proof?

Simon's Solution

I assumed that $-5 = 5$.

Liz started off with the false assumption that the two numbers were equal.

Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$

Everything that comes after the false assumption doesn't matter because the reasoning is built on the false assumption.

Even though $25 = 25$, the underlying premise is not true.

$-5 \neq 5$

Liz's conclusion is built on a false assumption, and the conclusion she reaches is the same as her assumption.

If an assumption is not true, then any argument that was built on the assumption is not valid.

Circular reasoning has resulted from these steps. Starting with an error and then ending by saying that the error has been proved is arguing in a circle.

circular reasoning
An argument that is incorrect because it makes use of the conclusion to be proved.

Your Turn

How is an error in a premise like a counterexample?

Answer

EXAMPLE 4 Using reasoning to determine the validity of a proof

Hossai is trying to prove the following number trick:
Choose any number. Add 3. Double it. Add 4. Divide by 2. Take away the number you started with.

Each time Hossai tries the trick, she ends up with 5. Her proof, however, does not give the same result.

Hossai's Proof

n	Choose any number.
$n + 3$	Add 3.
$2n + 6$	Double it.
$2n + 10$	Add 4.
$2n + 5$	Divide by 2.
$n + 5$	Take away the number you started with.

Where is the error in Hossai's proof?

Sheri's Solution

$1 \longrightarrow 5$
$10 \longrightarrow 5$

I tried the number trick twice, for the number 1 and the number 10. Both times, I ended up with 5. The math trick worked for Hossai and for me, so the error must be in Hossai's proof.

n ✔ The variable n can represent any number. This step is valid.

$n + 3$ ✔ Adding 3 to n is correctly represented.

$2n + 6$ ✔ Doubling a quantity is multiplying by 2. This step is valid. Its simplification is correct as well.

$2n + 10$ ✔ Adding 4 to the expression is correctly represented, and the simplification is correct.

$2n + 5$ ✘ The entire expression should be divided by 2, not just the constant. This step is where the mistake occurred.

I corrected the mistake:

$$\frac{2n + 10}{2} = n + 5$$
$$n + 5 - n = 5$$

I completed Hossai's proof by subtracting n. I showed that the answer will be 5 for any number.

EXAMPLE 5 Using reasoning to determine the validity of a proof

Jean says she can prove that $\$1 = 1¢$.

Jean's Proof

$1 can be converted to 100¢.
100 can be expressed as $(10)^2$.
10 cents is one-tenth of a dollar.
$(0.1)^2 = 0.01$
One hundredth of a dollar is one cent, so $1 = 1¢.

How can Jean's friend Grant show the error in her reasoning?

Grant's Solution

$1 can be converted to 100¢. ✔ It is true that 100 cents is the same as $1.

100 can be expressed as $(10)^2$. ✔ It is true that $(10)^2$ is $10 \cdot 10$, which is 100.

10 cents is one-tenth of a dollar. ✔ It is true that 10 dimes make up a dollar.

$(0.1)^2 = 0.01$ ✔ Arithmetically, I could see that this step was true. But Jean was ignoring the units. It doesn't make sense to square a dime. The units $¢^2$ and $\$^2$ have no meaning.

A dollar is equivalent to (10)($0.10) or 10(10¢), not to (10¢)(10¢) or ($0.10)($0.10).

$\$1 \neq 1¢$

In Summary

Key Idea

- A single error in reasoning will break down the logical argument of a deductive proof. This will result in an invalid conclusion, or a conclusion that is not supported by the proof.

Need to Know

- Division by zero always creates an error in a proof, leading to an invalid conclusion.
- Circular reasoning must be avoided. Be careful not to assume a result that follows from what you are trying to prove.
- The reason you are writing a proof is so that others can read and understand it. After you write a proof, have someone else who has not seen your proof read it. If this person gets confused, your proof may need to be clarified.

HOMEWORK...

p. 42: #1 - 10 (omit #8)

Mr. Svarc's Missing $ Problem...REALLY???

Two men were selling Atlantic Salmon Flies: one man sold 3 flies per dollar and the other man sold 2 flies per dollar.

One day they were both away so they each left 30 flies with a friend. To simplify the reckoning, the friend decided to sell 5 flies for 2 dollars. They sold them all and took in 24 dollars.

When it came to dividing up the sales between the owners...a problem arose. The one who had 30 flies at 3 for a dollar wanted $10. The other who had 30 flies at 2 for a dollar wanted $15. In total this made $25.

The friend only made $24 which means that they are a dollar short.

WHAT HAPPENED TO THE MISSING DOLLAR???

The 'Missing Dollar' Riddle...

[is a famous riddle that invoves an informal fallacy]

Three guests check into a hotel room. The clerk says the bill is $30, so each guest pays $10. Later the clerk realizes the bill should only be $25. To rectify this, he gives the bellhop $5 to return to the guests. On the way to the room, the bellhop realizes that he cannot divide the money equally. As the guests didn't know the total of the revised bill, the bellhop decides to just give each guest $1 and keep $2 for himself. Each guest got $1 back: so now each guest only paid $9; bringing the total paid to $27. The bellhop has $2. And $27 + $2 = $29 so, if the guests originally handed over $30, what happened to the remaining $1?

Solution:

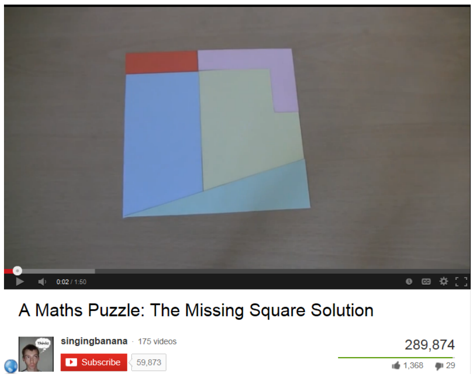
Start with the money that the clerk has...$25

Now add the change given to each person... 3 x $1

Finally add the bellhop tip...$2

TOTAL = $30 (there was NO missing dollar!)

<u>**OR**</u> 9 x $3 **SUBTRACT** $2 = $25



1.6 Reasoning to Solve Problems

GOAL

Solve problems using inductive or deductive reasoning.

EXPLORE...

- Suppose that you are lost in the woods for hours and come upon a cabin. In the cabin, you find a lantern, a candle, a wood stove with wood in it, and a match. What do you light first?

SAMPLE ANSWER

I would light the match first. If I didn't, I couldn't light any of the other items. I would light the candle next, since it would stay lit for longer than the match and would allow me to light the other two items. Also, it's less likely that I would make an error or fail when lighting the candle. The lantern and the stove would be more difficult to light.

INVESTIGATE the Math

Emma was given this math trick:
- Choose a number.
- Multiply by 6.
- Add 4.
- Divide by 2.
- Subtract 2.

Emma was asked to use inductive reasoning to make a conjecture about the relationship between the starting and ending numbers, and then use deductive reasoning to prove that her conjecture is always true. Here is her response to the problem:

Inductive reasoning:

#	×6	+4	÷2	−2
5	30	34	17	15
−3	−18	−14	−7	−9
0	0	4	2	0
24	144	148	74	72

I followed the steps to work through four examples.

Conjecture: It is 3 times.

Deductive reasoning:

I chose d.

Then I multiplied, added, divided, and subtracted to get an expression.

$\left(\frac{6d + 4}{2}\right) - 2$

It worked.

It simplified to $3d$.

How can Emma's communication about her reasoning be improved?

A. With a partner, explain why Emma might have chosen the values she did.

B. What details are missing from the deductive reasoning Emma used to arrive at the expression $3d$?

C. Improve Emma's conjecture, justifications, and explanations.

Answers

A.

B.

C.

Reflecting

D. How does it help to understand the mathematics when both symbols and words are used in an explanation?

E. Why is it important to explain your reasoning clearly?

Answers

D.

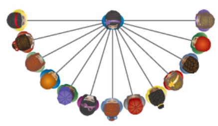
E.

APPLY the Math

EXAMPLE 1 Using reasoning to solve a problem

The members of a recently selected varsity basketball team met each other at their first team meeting. Each person shook the hand of every other person. The team had 12 players and 2 coaches. How many handshakes were exchanged?

Kim's Solution



I decided to think about how many times each person shook hands. There were 14 people in total, so person 1 shook hands with each of the other 13 people.

13 handshakes

Person 2 had already shaken hands with person 1. Person 2 shook hands with each of the remaining 12 people.

13 + 12 handshakes

13 + 12 + 11 + 10 + 9 + 8 + 7
+ 6 + 5 + 4 + 3 + 2 + 1
= 91 handshakes

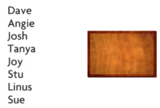
This pattern of handshakes continued until there were two people left when the last handshake happened.

EXAMPLE 2 Using reasoning to solve a problem

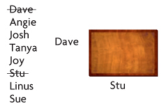
Sue signed up for games at her school's fun night. Seven other people were assigned to her group, making up four pairs of partners. The other members of her group were Dave, Angie, Josh, Tanya, Joy, Stu, and Linus. When the games started, Dave and his partner were to the left of Stu. Across from Dave was Sue, who was to the right of Josh. Dave's brother's partner, Tanya, was across from Stu. Joy was not on Stu's right.

Name the four pairs of partners.

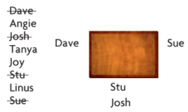
Vicky's Solution



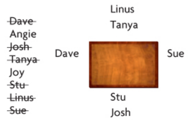
I drew a rectangle to represent a table. I made a list of the students' names so I could cross them off as I put them in place.



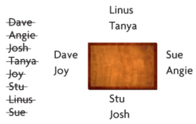
The first names I wrote in were Dave and Stu, since they were the first two mentioned. It didn't matter where I started, as long as I kept the relationships of left, right, and across the table. I crossed Dave and Stu off my list.



I knew that Sue was across from Dave and to the right of Josh. I crossed Sue and Josh off my list.



The next clue mentioned that Dave's brother and his partner Tanya were across from Stu. The only male name left was Linus, so Linus and Tanya were partners. I crossed their names off my list.



If Joy was not on Stu's right, then she must have been on his left. Therefore, she must have been Dave's partner. So, the last person to match was Angie with Sue.

The four pairs of partners were Linus and Tanya, Dave and Joy, Sue and Angie, and Stu and Josh.

The partners sat together, on the same side of the table.

Your Turn

Discuss, with a partner, whether Kim used inductive or deductive thinking in her solution. How do you know?

Answer

Your Turn

Discuss with a partner whether inductive or deductive reasoning was used for this solution. How do you know?

Answer

In Summary

Key Idea

- Inductive and deductive reasoning are useful in problem solving.

Need to Know

- Inductive reasoning involves solving a simpler problem, observing patterns, and drawing a logical conclusion from your observations to solve the original problem.
- Deductive reasoning involves using known facts or assumptions to develop an argument, which is then used to draw a logical conclusion and solve the problem.

HOMEWORK...

p. 48: #1 - 13
(OMIT #5, 8, 10, 11)

WARM UP PROBLEM: Need 4 gallons using only a 3 and 5 gallon jugs???

SOLUTIONS...

Step 1. Fill 5 gallon jug

Step 2. Pour 5 gallon jug into 3 gallon jug, leaving 2 remaining gallons in 5 gallon jug.

Step 3. Empty 3 gallon jug.

Step 4. Pour 2 gallons from 5 gallon jug into 3 gallon jug, leaving 1 gallon of empty space.

Step 5. Refill 5 gallon jug.

Step 6. Pour water from 5 gallon jug into 3 gallon jug, which already has 2 gallons in it, and only 1 gallon of empty space, leaving exactly 4 gallons in the 5 gallon jug.

there is an alternate way to solve this:

1. fill the 3 gallon jug
2. pour that 3 gallons into the 5 gallon jug
3. refill the 3 gallon jug
4. fill the 5 gallon jug to the top, leaving 1 gallon in the 3 gallon jug
5. empty the 5 gallon jug
6. pour the 1 gallon from the 3 gallon jug into the 5 gallon jug
7. refill the 3 gallon jug
8. pour that 3 gallons into the 5 gallon jug which already has 1 gallon in it for a total of 4 gallons.

1.7 Analyzing Puzzles and Games

GOAL

Determine, explain, and verify a reasoning strategy to solve a puzzle or to win a game.

EXPLORE...

• Three students are playing a game. Two of the students flip a coin, and the third student records their scores. Student 1 gets a point if the result is two heads, student 2 gets a point if the result is two tails, and student 3 gets a point if the result is a head and a tail. The first student to get 10 points wins. Explain whether you would prefer to be student 1, student 2, or student 3.

SAMPLE ANSWER

INVESTIGATE the Math

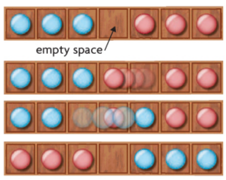
To solve a leapfrog puzzle, coloured counters are moved along a space on a board. The goal is to move each set of coloured counters to the opposite side of the board.

Board at start

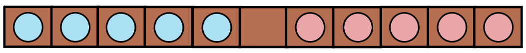
A counter can move into the empty space.

A counter can leapfrog over another counter into the empty space.

Board at end



❓ What is the minimum number of moves needed to switch five counters of each colour?



A. Develop a group strategy to switch the blue and red counters using as few moves as possible.

B. Execute your strategy, counting each move you make.

C. How many moves did you need to complete the switch?

Answers

A.

C.

Reflecting

D. How did you know that you had completed the switch in the fewest number of moves?

E. Did you use inductive or deductive reasoning to solve the puzzle? Explain.

F. Predict the minimum number of moves needed to solve the puzzle if you had six counters of each colour. Explain how you made your prediction.

G. Did you use inductive or deductive reasoning in step F? Explain.

Answers

D.

E.

F.

G.

APPLY the Math

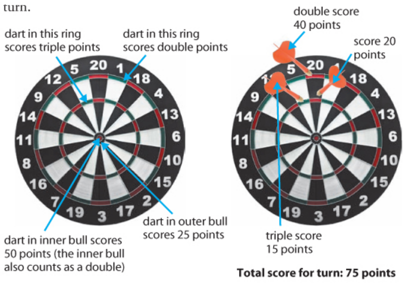
EXAMPLE 1 Using reasoning to determine possible winning plays

Frank and Tara are playing darts, using the given rules. Their scores are shown in the table below. To win, Frank must reduce his score to exactly zero and have his last counting dart be a double.

Rules

- Each player's score starts at 501.
- The goal is to reduce your score to zero.
- Players alternate turns.
- Each player throws three darts per turn.

Frank		Tara	
Turn Score	Total Score	Turn Score	Total Score
	501		501
100	401	85	416
95	306	85	331
140	166	140	191
130	36	91	100



What strategies for plays would give Frank a winning turn?

Frank's Solution

$2(18) = 36$ — I could win with a single dart in double 18.

$18 + 2(9) = 36$ — If I hit 18 instead of double 18, then I could use my second dart to try for double 9.

$18 + 9 = 27$
9 would be left. — If I hit 9 instead of double 9 with my second dart, then I couldn't win this turn. That's because I can't score 9 with a double.

Your Turn

a) Describe two other ways that Frank could win the game on his turn.

b) If Frank does not win on his turn, describe a strategy that Tara could use to win on her next turn.

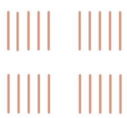
Answers

EXAMPLE 2 Using deductive and inductive reasoning to determine a winning strategy

Nadine and Alice are playing a toothpick game. They place a pile of 20 toothpicks on a desk and alternate turns. On each turn, the player can take one or two toothpicks from the pile. The player to remove the last toothpick is the winner. Nadine and Alice flip a coin to determine the starting player.

Is there a strategy Alice can use to ensure that she wins the game?

Try the game by dragging toothpicks into Nadine's and Alice's play areas.



Nadine | Alice

Alice's Solution

I need to make sure that there are one or two toothpicks left after Nadine's last turn. — I will win the game if I can take the last toothpick. If I work backward, I might see a pattern I can use to win.

To make sure this happens, I have to leave three toothpicks on the desk for Nadine. — If I leave three toothpicks, Nadine has to take either one or two toothpicks. If she takes only one, I can take the two that are left and win. If she takes two, I can take the last one and win.

To make sure this happens, I have to leave six toothpicks on the desk for Nadine. — If I leave six toothpicks, Nadine has to take either one or two toothpicks. If she takes only one, I can take two, which would leave three. If she takes two, I can take one and leave her with three.

To make sure this happens, I have to leave nine toothpicks on the desk for Nadine. — If I leave nine toothpicks, Nadine has to take either one or two toothpicks. If she takes only one, then I can take two, which would leave six. If she takes two, I can take one and leave her with six.

I can see that I need to leave 12, 15, and 18 toothpicks for Nadine. — There is a pattern to the number of toothpicks I must leave for Nadine: 3, 6, 9, 12, 15, 18.

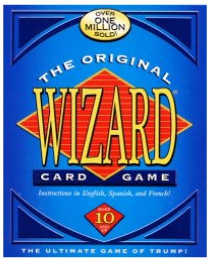
I will win if I go first and take two toothpicks. Each turn after that, I need to pick one or two so that I leave Nadine with a number of toothpicks that is a multiple of 3. — If Nadine goes first and knows this strategy, I can't win. If she goes first and doesn't know this strategy, however, I can win by arranging to leave her a number of toothpicks that is a multiple of 3.

Your Turn

a) Which part of Alice's strategy involved deductive reasoning? Explain.

b) Which part of Alice's strategy involved inductive reasoning? Explain.

Answers



Wizard Game...

Intro. to the cards

What is a trick???

What is trump???

In the first round every player gets one card. In the subsequent rounds the number of cards is increased by one until all cards are distributed. That means that three players play 20 rounds, four players 15 rounds, five players 12 rounds and six players 10 rounds. The top card of the remaining cards is turned over to determine the trump color. If there are no cards left or a jester is turned there is no trump color only the wizards are trump. If a wizard is turned the dealer picks a trump color.

After looking at their cards, starting with the player to the dealer's left, each player states how many tricks he believes he will take, from zero to the number of cards dealt. This is recorded on a score pad.

The player to the left of the dealer plays a card and then the others follow clockwise. If a card other than a wizard or jester is played the players have to follow suit, but it is possible to play a jester or wizard although the player has the desired suit. If a suit is played but then a Wizard is played, the next players must follow the card initially played if they have that suit. The Wizard beats all other cards but the first one in a trick beats all others. The jester is beaten by all others, but if all cards in a trick are jesters the first one beats the others. If a jester is played as the first card the first suit card decides which suit has to be followed. If a wizard is played as the first card every player is free to play what they want regardless of the others.

At the end of each round, each player is given a score based on his performance. For predicting the number of tricks taken correctly, a player receives 20 points plus 10 points for each trick taken. For predicting the number of tricks taken incorrectly, a player loses 10 points for each trick over **or** under.

In Summary

Key Idea

- Both inductive reasoning and deductive reasoning are useful for determining a strategy to solve a puzzle or win a game.

Need to Know

- Inductive reasoning is useful when analyzing games and puzzles that require recognizing patterns or creating a particular order.
- Deductive reasoning is useful when analyzing games and puzzles that require inquiry and discovery to complete.

Review Chapter 1...

Test is on Wednesday

1.1: Terms - conjecture, inductive reasoning
- develop a conjecture
- evaluate a conjecture

1.2: - testing a conjecture (valid vs invalid)
- through measurement

1.3: Term - counterexample
- find a counterexample
- evaluate a counterexample

1.4: Terms - deductive reasoning
- algebra to prove the conjecture
- evaluate a proof

1.5: Terms: premise, generalization, circular reasoning
- find errors
- correct errors
- 'common errors'... 1) false assumptions
  2) dividing by zero
  3) calculation errors
  4) logical errors

1.6: - solve a problem using logic
- strategies: pictures, charts, algebra, etc..
- using patterns/examples

1.7: - strategies with gaming/puzzles

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1s1e1 finalt highquality.mp4

1s1e2 finalt.mp4

1s1e4 finalt.mp4

1s3e3 final.mp4

three_dragons.wmv

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1s5e1 finalt.mp4

1s5e3 finalt.mp4

1s6e1 finalt.mp4

1s6e2 final.mp4

1s7e1finalt2.mp4

1s7e2 finalt.mp4