Warm Up

Given that $f(x) = -2x^2 + 5x - \sqrt{x}$, determine the value of…

(1) $f(4)$ (2) $f(\$)$ (3) $f(9+h)$

$= -2(4)^2 + 5(4) - \sqrt{4}$

$= -32 + 20 - 2$

$= -14$

$= -2\$^2 + 5\$ - \sqrt{\$}$

(3) $f(9+h) = -2(9+h)^2 + 5(9+h) - \sqrt{9+h}$

$= -2(81 + 18h + h^2) + 45 + 5h - \sqrt{9+h}$

$= -117 - 31h - 2h^2 - \sqrt{9+h}$

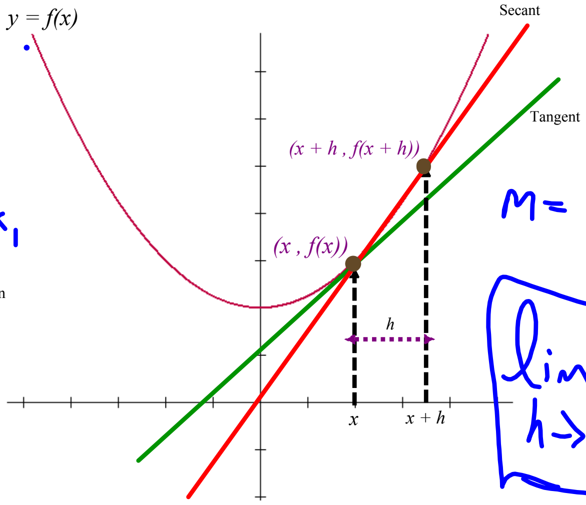
~~Tangents~~, Velocities, and Rates of Change

Slope of a tangent to a curve:

slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

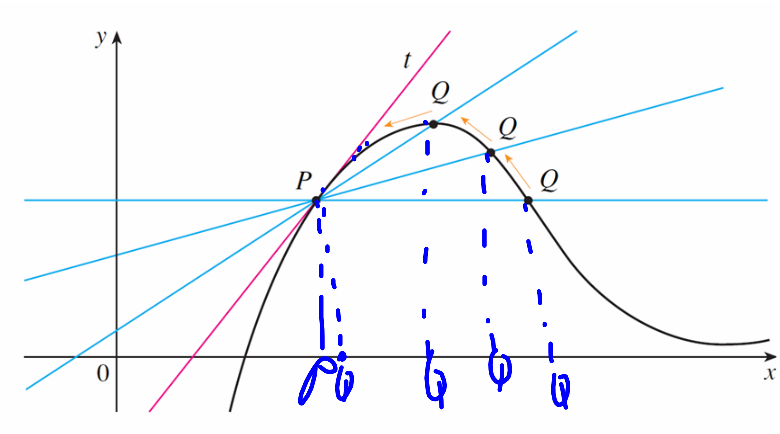
How will we find the slope of a tangent drawn to a curve at a point *x*?



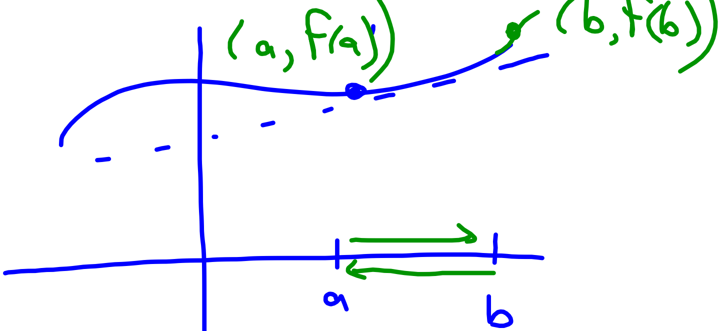
$$m = \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

How will the slope of this secant become a better approximation for the slope of the tangent line?



Use your knowledge of limits to determine an expression for that would represent the slope of the tangent line drawn at the point *x*.



$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Example:

Determine the equation of the tangent line drawn to the curve $y = x^2 + 2$ at the point $x = -2$.

point & slope

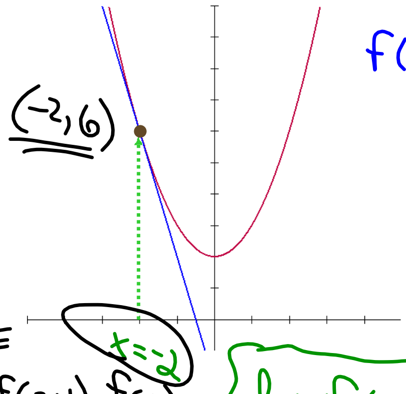
$y - y_1 = m(x - x_1)$

$y = mx + b$

$f(x) = x^2 + 2$

$f(x+h) = (x+h)^2 + 2$

$= x^2 + 2xh + h^2 + 2$



or

$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 2) - (x^2+2)}{h}$

$\lim_{h \to 0} \frac{2xh + h^2}{h}$

$\lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$\boxed{m = 2x}$

Slope at $x = -2$

$m = 2(-2)$

$m = -4$

Equation:

$y - y_1 = m(x - x_1)$

$y - 6 = -4(x + 2)$

$y = mx + b$

or $Ax + By + C = 0$ Standard Form

$A > 0$

A, B, C can Not be fractions

$y - 6 = -4x - 8$

$\boxed{y = -4x - 2}$

or

$\boxed{4x + y + 2 = 0}$

Find equation of the tangent to

$f(x)=\sqrt{x+7}$ at $x=2$

slope

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(x+h)=\sqrt{x+h+7}$

$$\lim_{h\to 0}\frac{\sqrt{x+h+7}-\sqrt{x+7}}{h}\left(\frac{\sqrt{x+h+7}+\sqrt{x+7}}{\sqrt{x+h+7}+\sqrt{x+7}}\right)$$

$$\lim_{h\to 0}\frac{\cancel{(x+h+7)}-\cancel{(x+7)}}{\cancel{h}\left(\sqrt{x+h+7}+\sqrt{x+7}\right)}$$

$$m=\frac{1}{2\sqrt{x+7}}$$

at $x=2$ ...

$f(x)=\sqrt{x+7}$

$m=\frac{1}{2\sqrt{2+7}}$

$f(2)=\sqrt{2+7}$

$=3$

$m=\frac{1}{6}$

$(2,3)$

$y=mx+b$

$3=\frac{1}{6}(2)+b$ (6)

$18=2+6b$

$\frac{16}{6}=\frac{6b}{6}$

$\frac{8}{3}=b$

$y=\frac{1}{6}x+\frac{8}{3}$

Homework:

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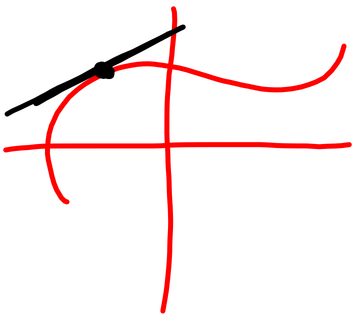
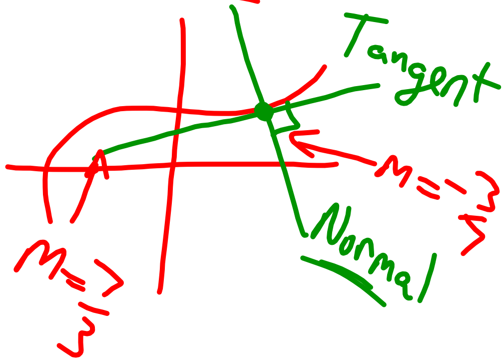
#7 (i), (ii), (iv) and (v)

Tangent

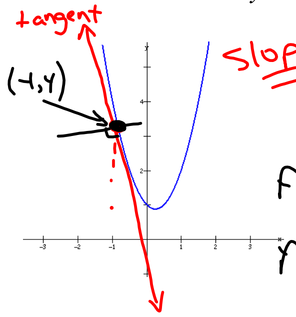


Normal



Warm Up

Determine the equation of the NORMAL drawn to the curve $y = 2x^2 - x + 1$ at $x = -1$.



slope $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = 2x^2 - x + 1$

$f(x+h) = 2(x+h)^2 - (x+h) + 1$

$= 2(x^2 + 2xh + h^2) - x - h + 1$

$= 2x^2 + 4xh + 2h^2 - x - h + 1$

$$\lim_{h \to 0} \frac{(\cancel{2x^2} + 4xh + 2h^2 \cancel{- x} - h \cancel{+ 1}) - (\cancel{2x^2} \cancel{- x} \cancel{+ 1})}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 1)}{\cancel{h}}$$

$\boxed{m = 4x - 1}$

$\therefore$ at $x = -1$

$m = 4(-1) - 1 = \frac{-5}{1}$ } Tangent

$\therefore$ Normal

$m = \frac{1}{5}$

Point: $x = -1$

$y = 2x^2 - x + 1$

$y = 2(-1)^2 - (-1) + 1$

$= 2 + 1 + 1$

$= 4$

$(-1, 4)$

$y - y_1 = m(x - x_1)$

$(5)\, y - (5)\, 4 = (\cancel{5}) \frac{1}{\cancel{5}}(x + 1) \Rightarrow$ (General) Standard Form

$5y - 20 = x + 1$

$0 = x - 5y + 21$

$y - 4 = \frac{1}{5}x + \frac{1}{5}$

$y = mx + b$

$y = \frac{1}{5}x + \frac{1}{5} + \frac{4}{1}$

$y = \frac{1}{5}x + \frac{21}{5}$

Slope y-Intercept Form

g) $\lim_{u \to \infty} \frac{2u^4 - u^2 - 3}{(u^2-4)(2u^2+5)}$

$\lim_{u \to \infty} \frac{\frac{2\cancel{u^4}}{\cancel{u^4}} - \frac{u^2}{u^4} - \frac{3}{u^4}}{\frac{2\cancel{u^4}}{\cancel{u^4}} + \frac{5u^2}{u^4} - \frac{8u^2}{u^4} - \frac{20}{u^4}}$

$= \frac{2-0-0}{2+0-0-0} = \frac{2}{2} = 1$

(j) $\lim_{x \to -\infty} 5^{3-x}$

$= 5^{3-(-\infty)}$

$= 5^{\infty}$

$\rightarrow \infty$

$\therefore$ DNE

$5^{3-\infty}$

$5^{-\infty}$

$\frac{1}{5^{\infty}}$

$= 0$

b) $\lim_{x \to 4} \dfrac{3x^2 - 10x - 8}{5x - 1} = \dfrac{0}{19}$

$= 0$

2. $f(x) = \begin{cases} x^2 - 1, & x < 2 \\ 5 - x, & 2 \le x < 4 \\ -1 & x = 4 \\ 2x - 7 & x > 4 \end{cases}$

① $y = x^2 - 1$

$V(0, -1)$

x	y
2	3

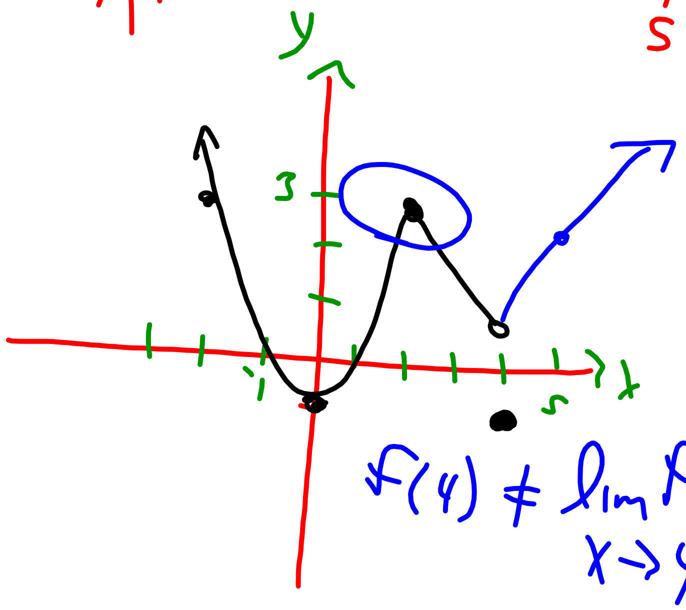
② $y = 5 - x$

x	y
2	3
4	1

③ $(4, -1)$

③ $y = 2x - 7$

x	y
4	1
5	3



$f(4) \neq \lim_{x \to 4} f(x)$

Derivatives

f' ← prime

*The concept of a **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change).*

2 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

slope of a tangent →

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

← Also known as the "First Principle of Calculus"

if this limit exists.

...or this definition can also be expressed as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Most common notations used to represent the derivative...

$$f'(x) \quad \& \quad \frac{dy}{dx} \qquad \frac{\Delta y}{\Delta x}$$

Leibniz Notation

$f''(x)$ ← second derivative

d

Notation:

$$f'(x) \Leftrightarrow \frac{dy}{dx}$$

$$f''(x) \Leftrightarrow \frac{d^2y}{dx^2} \qquad \frac{d^5y}{dx^5}$$

Examples:
Use the definition of a derivative to differentiate...

(1) $f(x) = 2x^2 - 3x + 1$ $\quad = 4x - 3$ (2) $y = \sqrt{x+2}$

ex.

2/ $y = \sqrt{x+2}$ $\quad y' = \frac{1}{2}(x+2)^{-\frac{1}{2}}$

Definition: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\qquad \lim_{x \to 5} \sqrt{5-x}$

$f(x) = \sqrt{x+2}$

$f(x+h) = \sqrt{x+h+2}$

$$\lim_{h \to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right)$$

$$\lim_{h \to 0} \frac{(x+h+2) - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$\lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$f'(x) = \frac{1}{\sqrt{x+2} + \sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

at $x = 7$

Test:

1. a) $\frac{1}{6}$

b) $\frac{16}{3}$

c) $\lim_{x \to 10} \sqrt[3]{2-x}$

$= \sqrt[3]{2-10}$

$= -2$

d) $\lim_{x \to -1} \frac{3x+\sqrt{x+10}}{x^2-1}\left(\frac{3x-\sqrt{x+10}}{3x-\sqrt{x+10}}\right)$

$\lim_{x \to -1} \frac{\overset{\cancel{(x+1)}(9x-10)}{9x^2-(x+10)}}{(x-1)\cancel{(x+1)}(3x-\sqrt{x+10})}$

$9x^2-x-10 \quad -90$

$9x^2-10x+9x-10$

$x(9x-10)+1(9x-10)$

$= \frac{-19}{(-2)(-3-3)}$

$= \frac{-19}{12}$

e) $\lim_{x \to 2a} \frac{\left[\cancel{(2x+a)-5a}\right]\left[(2x+a)^2+5a(2x+a)+25a^2\right]}{\cancel{(x-2a)}(x+2a)}$

$2x-4a$

$2\cancel{(x-2a)}$

$= \frac{2\left[(4a+a)^2+5a(4a+a)+25a^2\right]}{(2a+2a)}$

$= \frac{2\left[(25a^2)+25a^2+25a^2\right]}{4a}$

$= \frac{150a^2}{4a}$

$= \frac{75a}{2}$

(f) $\frac{-1}{16}$

(g) $\frac{5}{24}$

h) $\lim_{x\to 1} \sqrt{\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x^2-1)}(x^2+1)}}$ $\cancel{(x-1)}(x+1)$

$= \sqrt{\frac{3}{(2)(2)}}$

$= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

i) $\lim_{x\to -7} \frac{|x+7|}{(x-7)(x+7)}$

$-8 \quad -7 \quad -6$

$\lim_{x\to -7^-} \frac{\cancel{|-7.000..1+7|}^{-1}}{(-7.00..-7)\cancel{(-7.00..1+7)}}$ $= \frac{1}{14}$

$\lim_{x\to -7^+} \frac{\cancel{|-6.99..+7|}^{1}}{(-6.99..-7)\cancel{(-6.99..+7)}}$ $= \frac{1}{-14}$

$\therefore \lim_{x\to -7} f(x)$ D.N.E

J) $\lim_{x\to -\infty} \frac{5^x}{7}$

$= \frac{5^{-\infty}}{7} = \frac{\left(\frac{1}{5^\infty}\right)}{7}$ $\quad \frac{0}{7}$

$= \frac{1}{7(5^\infty)}$ $\quad = 0$

$= \frac{1}{\infty}$

$\to 0$

k) $\lim_{x\to 2} \frac{\cancel{(x^2-4)}(x^4+4x^2+16)}{(x-12)\cancel{(x+2)}}$ $(x-2)\cancel{(x+2)}$

$= \frac{(0)(...)}{(2-12)}$

$\frac{0}{-10}$

0

l) $\lim_{x\to 0} \frac{\sin^7 3x}{5x^8-3x^7}$

$\lim_{x\to 0} \frac{\sin^7 3x}{x^7(5x-3)}$

$\lim_{x\to 0} \left(\frac{\sin 3x}{3x}\right)^7 \frac{1(3)^7}{5x-3}$

$(1)^7\left(\frac{2187}{0-3}\right)$

$= -729$

2/ $\underline{x=-1}$

continuous

$\underline{x=2}$

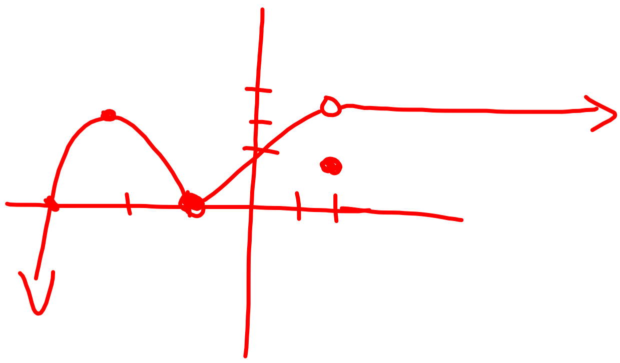
$f(2)=1$

$\lim_{x\to 2^-} f(x) = 3$ $\lim_{x\to 2^+} f(x) = 3$

$\lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ discontinuous

b)



#3/ a) -1 b) DNE

c) 0 d) -6

e) -6 f) -2

g) DNE h) -6

Bonus

$$\lim_{x\to 1} \frac{x^{\frac{1}{6}}-1}{x^{1/4}-1}$$

$$= \frac{\cancel{(x^{1/12}-1)}(x^{1/12}+1)}{\cancel{(x^{1/12}-1)}(x^{2/12}+x^{1/12}+1)}$$

$$= \frac{1+1}{1+1+1} = \frac{2}{3}$$

Example:

Determine the equation of a tangent drawn to the curve $f(x)=\frac{2}{1-3x}$ at $x=1$.

Remember that the equation of a line is found by using the point-slope formula... $y-y_1=m(x-x_1)$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x)=\frac{2}{1-3x}$$

$$f(x+h)=\frac{2}{1-3(x+h)}=\frac{2}{1-3x-3h}$$

$$\lim_{h\to 0}\frac{\frac{2}{1-3x-3h}-\frac{2}{1-3x}}{h}$$

$$\lim_{h\to 0}\left[\frac{2(1-3x)-2(1-3x-3h)}{(1-3x-3h)(1-3x)}\right]\cdot\frac{1}{h}$$

$$\lim_{h\to 0}\frac{\cancel{2}\cancel{-6x}\cancel{-2}\cancel{+6x}+6\cancel{h}}{(1-3x-3h)(1-3x)}\cdot\frac{1}{\cancel{h}}$$

$$m=\frac{6}{(1-3x)(1-3x)}$$

$$y'=\frac{6}{(1-3x)^2}\ \ldots$$

At $x=1$

Slope $m=\frac{6}{(1-3(1))^2}=\frac{6}{4}$

$m=\frac{3}{2}$

Point of tangency

$$f(x)=\frac{2}{1-3x}$$

$$f(1)=\frac{2}{1-3(1)}=-1$$

$(1,-1)$

$$2(y+1)=(2)\frac{3}{\cancel{2}}(x-1)$$

$$2y+2=3x-3$$

$$0=3x-2y-5$$

3. Determine $f'(-2)$ given that $f(x)=\dfrac{x^2-1}{2x+3}$

$$f(x+h)=\frac{(x+h)^2-1}{2(x+h)+3}$$

$$=\frac{x^2+2xh+h^2-1}{2x+2h+3}$$

$$\boxed{\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$\lim_{h\to 0}\frac{\dfrac{x^2+2xh+h^2-1}{2x+2h+3}-\dfrac{x^2-1}{2x+3}}{h}$$

$$\lim_{h\to 0}\left[\frac{(2x+3)(x^2+2xh+h^2-1)-(x^2-1)(2x+2h+3)}{(2x+2h+3)(2x+3)}\right]\left(\frac{1}{h}\right)$$

$$\lim_{h\to 0}\frac{(\cancel{2x^3}+4x^2h+2xh^2\cancel{-2x}+\cancel{3x^2}+6xh+3h^2\cancel{-3})-(\cancel{2x^3}+2x^2h+\cancel{3x^2}\cancel{-2x}-2h\cancel{-3})}{(2x+2h+3)(2x+3)(h)}$$

$$\lim_{h\to 0}\frac{\cancel{h}(4x^2+2xh+6x+3h-2x^2+2)}{(2x+2h+3)(2x+3)\cancel{(h)}}$$

$$f'(x)=\frac{2x^2+6x+2}{(2x+3)^2}$$

$$f'(-2)=\frac{2(-2)^2+6(-2)+2}{(2(-2)+3)^2}$$

$$=\frac{8-12+2}{1}$$

$$\boxed{f'(-2)=-2}$$

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#4, 5, 10, 11

Extra...

Determine the equations of both lines passing through the point (2 , -3) that are tangent to the parabola $y = x^2 + x$.

Attachments

Worksheet - Sketching Angles in Radians.doc

Warm-Up - Intro to Limits.docx

Review - Factoring.pdf

Worksheet - Factoring Review.doc