Combination of Functions

- Two functions f and g can be combined to form new functions
  - $f+g$,
  - $f-g$,
  - fg, and
  - f/g

  just as we add, subtract, multiply, and divide real numbers.

- This is summarized in the following table:

Algebra of Functions Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows:

$(f+g)(x) = f(x) + g(x)$ domain $= A \cap B$ ← Intersection (overlap of sets)

$(f-g)(x) = f(x) - g(x)$ domain $= A \cap B$

$(fg)(x) = f(x)g(x)$ domain $= A \cap B$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $= \{x \in A \cap B \mid \underline{g(x) \neq 0}\}$

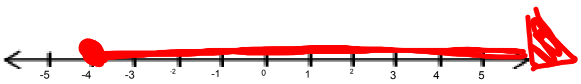
- ***Review of Intersection and Union of two sets:***

$f(x)=\sqrt{x+4}$ $g(x)=\sqrt{x^2-9}$

Let A represent the domain of f and B the domain of g.

A: $x+4 \geq 0$

$x \geq -4$



B: $x^2-9 \geq 0$

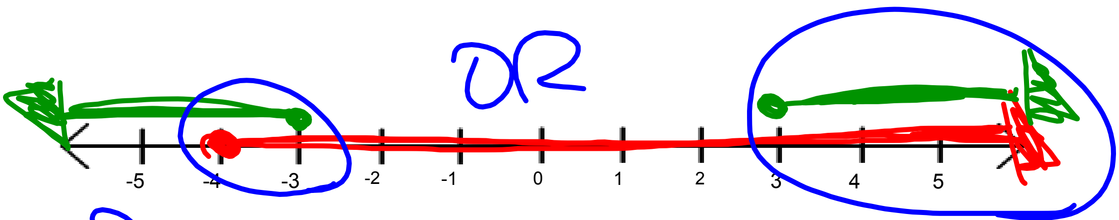
$x^2-9=0$

$(x-3)(x+3)=0$

$x=\pm 3$

$x \leq -3$ OR $x \geq 3$

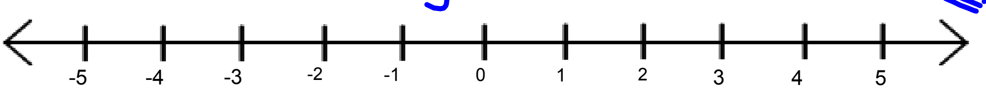
I. Intersection: $A \cap B$



$\{x \mid -4 \leq x \leq -3$ OR $x \geq 3, x \in R\}$

II. Union: $A \cup B$ Union of sets (Join together all from both)

$\{x \in R\}$



Example

- If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{4-x^2}$, find the functions $f+g$, $f-g$, fg, and f/g.

**Also examine the domain of each of these new functions

$(f+g)(x) = f(x)+g(x)$
$= \sqrt{x}+\sqrt{4-x^2}$

$(f-g)(x) = \sqrt{x} - \sqrt{4-x^2}$

$(\sqrt{2})(\sqrt{7})$
$\sqrt{2\cdot 7}$
$\sqrt{14}$

$(fg)(x) = (\sqrt{x})(\sqrt{4-x^2})$
$= \sqrt{4x-x^3}$

$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{4-x^2}} = \sqrt{\frac{x}{4-x^2}}$

$x \neq 2,-2 \longrightarrow$

Rationalize Denominator

$= \frac{\sqrt{x}}{\sqrt{4-x^2}}\left(\frac{\sqrt{4-x^2}}{\sqrt{4-x^2}}\right)$

$= \frac{\sqrt{4x-x^3}}{4-x^2}$

① $\frac{3}{\sqrt{7}}\left(\frac{\sqrt{7}}{\sqrt{7}}\right)$

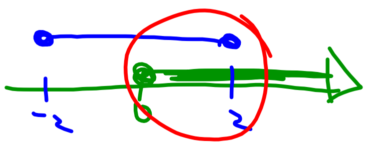
② $\frac{3}{2+\sqrt{7}}\left(\frac{2-\sqrt{7}}{2-\sqrt{7}}\right)$

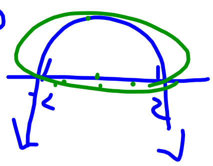
$f(x)=\sqrt{x}$

$g(x)=\sqrt{4-x^2}$

D: $x \geq 0$

D: $4-x^2 \geq 0$
$(2-x)(2+x)=0$
$x=\pm 2$





$0 \leq x \leq 2$

$-2 \leq x \leq 2$

Compositions of Functions

When the input in a function is another function, the result is called a **composite function.** If

$f(2)$

$f(x) = 3x + 2$ and $g(x) = 4x - 5$

then $f[g(x)]$ is a composite function. The statement $f[\underline{g(x)}]$ is read " f of g of x" or "the composition of f with g." $f[g(x)]$ can also be written as

$$(f \circ g)(x) \quad \text{or} \quad f \circ g(x) \quad \text{oR} \quad f[g(x)]$$

The symbol between f and g is a small open circle. When replacing one function with another, be very careful to get the order correct because compositions of functions are not necessarily commutative (as you'll see).

Example 1

If $f(x) = 3x + 2$ and $g(x) = 4x - 5$, find each of the following.

1. $f[g(4)]$
2. $g \circ f(4)$
3. $f[g(x)]$
4. $(g \circ f)(x)$

1. $f[g(4)] =$

$g(4) = 4(4) - 5$
$= 11$

$f(11) = 3(11) + 2$
$= 35$

2/ $g \circ f(4)$
$g[f(4)]$

$f(4) = 3(4) + 2 = 14$

$g(14) = 4(14) - 5$
$= 51$

3/ $f[g(x)]$

$3(g(x)) + 2$

$3(4x - 5) + 2$

$12x - 15 + 2$

$= 12x - 13$

4/ $(g \circ f)(x)$

$4(f(x)) - 5$

$4(3x + 2) - 5$

$12x + 8 - 5$

$= 12x + 3$

Example 2

If $f(x) = 3x^2 + 2x + 1$ and $g(x) = 4x - 5$, find each of the following:

1. $f[g(x)]$
2. $g[f(x)]$

$f[g(x)] = 3[g(x)]^2 + 2[g(x)] + 1$

$= 3(4x-5)^2 + 2(4x-5) + 1$

$= 3(16x^2 - 40x + 25) + 8x - 10 + 1$

$= \underline{48x^2 - 112x + 66}$

$g[f(x)] = 4[f(x)] - 5$

$= 4(3x^2 + 2x + 1) - 5$

$= 12x^2 + 8x - 1$

Check Up

Given the three functions....

$f(x) = 1 - x$ $g(x) = \sqrt{x+1}$ $h(x) = x^2 + 5$

Evaluate each of the following:

1. $(f \circ g)(3)$
2. $(g \circ h)(0)$
3. $(g \circ g \circ f)(-7)$
4. $(h \circ g \circ h \circ f)(-1)$
5. $(f \circ h \circ g)(m)$
6. $f\left(h\left(\sqrt{9x^4 - 1}\right)\right)$.

① $\Rightarrow f[g(3)]$

$g(3) = \sqrt{3+1}$
$= 2$
$f(2) = 1 - 2$
$= -1$

② $h(0) = 0^2 + 5$
$= 5$
$g(5) = \sqrt{5+1}$
$= \sqrt{6}$

3/ $f(-7) = 1 - (-7)$
$= 8$
$g(8) = \sqrt{8+1}$
$= 3$
$g(3) = \sqrt{3+1}$
$= \sqrt{4}$
$= 2$

4/ $f(-1) = 1 - (-1)$
$= 2$
$h(2) = (2)^2 + 5$
$= 9.$
$g(9) = \sqrt{9+1}$
$= \sqrt{10}$
$h(\sqrt{10}) = (\sqrt{10})^2 + 5$
$= 15$

5/ $g(m) = \sqrt{m+1}$
$h(\sqrt{m+1}) = (\sqrt{m+1})^2 + 5$
$= m + 1 + 5$
$= m + 6$
$f(m+6) = 1 - (m+6)$
$= -m - 5$

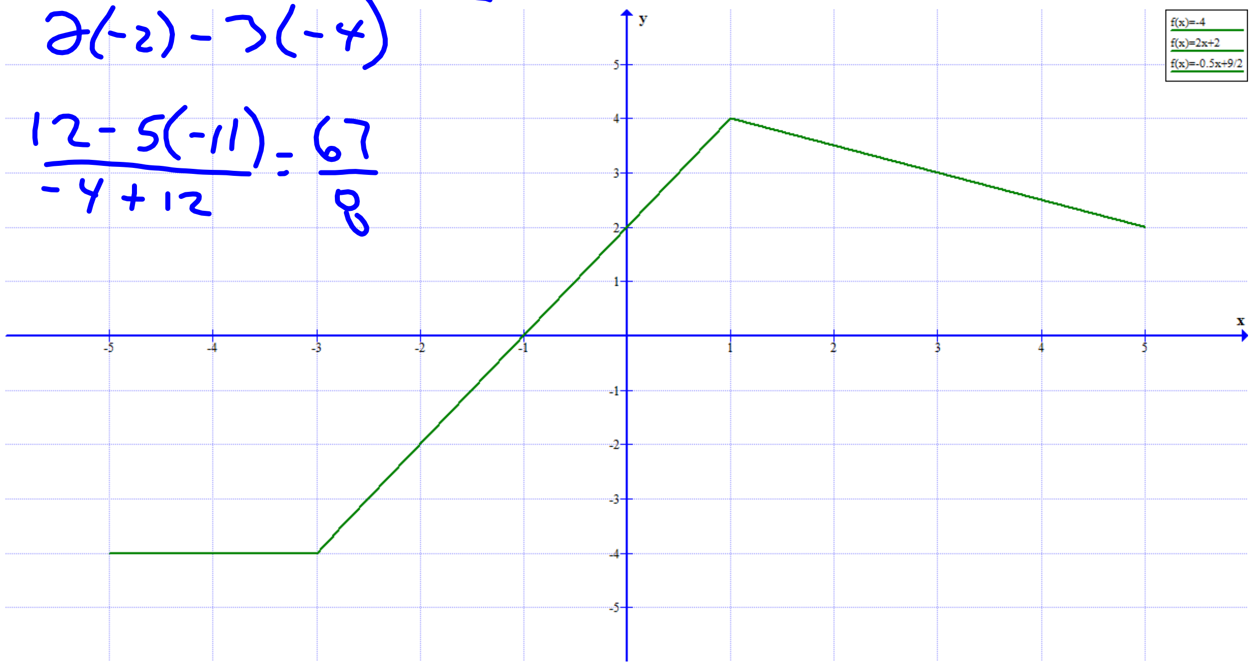
6/ $h(\sqrt{9x^4 - 1}) = (\sqrt{9x^4 - 1})^2 + 5$
$= 9x^4 + 4$
$f(9x^4 + 4) = 1 - (9x^4 + 4)$
$= -9x^4 - 3$

Given the graph of f(x) shown below, evaluate the following:

$$\frac{3f(1)-5\left[f(3)-7f(0)\right]}{2f(-2)-3f(-4)}$$

$$\frac{3(4)-5[3-7(2)]}{2(-2)-3(-4)}$$

$$\frac{12-5(-11)}{-4+12}=\frac{67}{8}$$



Review...

⇒ Sketch piecewise function
⇒ Function Notation
  ⇒ combinations: Domain (Intersection of Domains)
  ⇒ compositions:
⇒ Catalog of essential functions

Notes

Functions: Transformations

→ Translations, Reflections, Stretches

$$y = a f[b(x-h)] + k$$

- a: -Vert. stretch, -Reflect in x-axis
- b: - Reflect in y-axis, - Horizontal stretch $\left(\frac{1}{b}\right)$
- h: Horizontal Translation
- k: Vertical Translation

Mapping:

$$(x, y) \rightarrow \left(\frac{1}{b}x + h,\ ay + k\right)$$

⇒ Inverse Functions

- Switch "x" & "y" (Domain & Range)
- Sketch Inverses from a given graph (Reflects in line $y = x$)
- One-one function (Horizontal line)
- Switch to inverse algebraically

ie. $f(x) = x + 7$

$x = y + 7$

$x - 7 = y$

$f^{-1}(x) = x - 7$

Chapter Review from textbook...

Pages 56-57
#2, 3, 6, 8, 9, 10, 11, 14, 15, 16

Practice Test
Pages 58-59
All questions