Practice Exercises:

Determine each of the following areas using a Riemann Sum with an infinite number of rectangles:

(1) Area below $f(x) = 2x^2 - 1$ between $x = 1$ and $x = 3$. $\frac{46}{3}$

(2) Area below $f(x) = x^2 - 4x + 7$ between $x = -1$ and $x = 2$. $\frac{10}{3}$

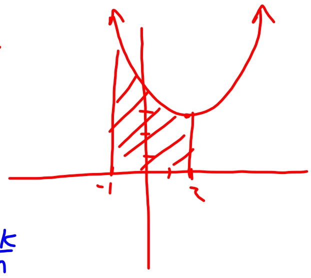
(3) Area below $f(x) = x^3 + 2x^2 + x$ between $x = 0$ and $x = 1$. $\frac{17}{12}$

$y = x^2 - 4x + 7$ $\quad x = -1$ & 2

$y = (x^2 - 4x + \underline{4}) + 7 - 4$

$y = (x-2)^2 + 3$

$V(2,3)$



$\Delta x = \frac{3}{n} \quad x_k = -1 + \frac{3k}{n}$

$A = \Delta x \sum_{k=1}^{n} f(x_k)$

$A = \frac{3}{n} \sum_{k=1}^{n} f\left(-1 + \frac{3k}{n}\right) \qquad f(x) = x^2 - 4x + 7$

$A = \frac{3}{n} \sum_{k=1}^{n} \left(-1 + \frac{3k}{n}\right)^2 - 4\left(-1 + \frac{3k}{n}\right) + 7$

$A = \frac{3}{n} \sum_{k=1}^{n} \left(1 - \frac{6k}{n} + \frac{9k^2}{n^2} + 4 - \frac{12k}{n} + 7\right)$

$A = \frac{3}{n} \sum_{k=1}^{n} \left(\frac{9k^2}{n^2} - \frac{18k}{n} + 12\right)$

$A = \frac{3}{n} \left[\frac{9}{n^2} \sum_{k=1}^{n} k^2 - \frac{18}{n} \sum_{k=1}^{n} k + \sum_{k=1}^{n} 12\right)$

$A = \frac{3}{n} \left[\frac{9}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) - \frac{18}{n}\left(\frac{n^2}{2} + \frac{n}{2}\right) + 12n\right]$

$A = \frac{3}{n}\left(3n + \frac{9}{2} + \frac{3}{2n} - 9n - \frac{9}{1} + 12n\right)$

$A = \frac{3}{n}\left(6n + \frac{3}{2n} - \frac{9}{2}\right)$

$A = 18 + \frac{9}{2n^2} - \frac{27}{2n}$

$\lim_{n \to \infty} \left(18 + \frac{9}{2n^2} - \frac{27}{2n}\right)$

$= 18u^2$

Refresher...

Determine the general antiderivative:

$5x^4$ $\sqrt[5]{x^7}$

$x^{1/2}$ $\frac{1}{2}x^{-\frac{1}{2}}$ $x^3 e$

$$f'(x)=\left(\frac{5x}{(7x^2-2)^4}-\frac{\csc\sqrt{x}\cot\sqrt{x}}{\sqrt{x}}+\frac{\sin 2x}{\sqrt{1-\cos^2(2x)}}-x^3e^{5x^4}+\frac{3x-2}{3x^2-4x+1}+\sqrt[5]{x^7}-8\pi\right)dx$$

$$f'(x)=\frac{5}{14}(7x^2-2)^{-4}14x-(2)\csc\sqrt{x}\cot\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)+\frac{-1}{2}\frac{-\sin(2x)(2)}{\sqrt{1-(\cos 2x)^2}}$$

$$+\frac{1}{2}\left(\frac{6x-4}{3x^2-4x+1}\right)-\frac{1}{20}e^{5x^4}(20x^3)+x^{7/5}-8\pi$$

$$f(x)=-\frac{5}{42}(7x^2-2)^{-3}+2\csc\sqrt{x}-\frac{1}{2}\sin^{-1}(\cos 2x)$$

$$+\ln|3x^2-4x+1|-\frac{1}{20}e^{5x^4}+\frac{5}{12}x^{12/5}-8\pi x+C$$

Definite Integral

2 Definition of a Definite Integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Remarks

- In the notation $\int_a^b f(x)dx$...
  - the symbol $\int$ is called an *integral sign* and resembles a stretched-out "S";
  - $f(x)$ is called the *integrand* and a and b are the *upper* and *lower limits of integration*, resp.
  - we could replace x with any other letter without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

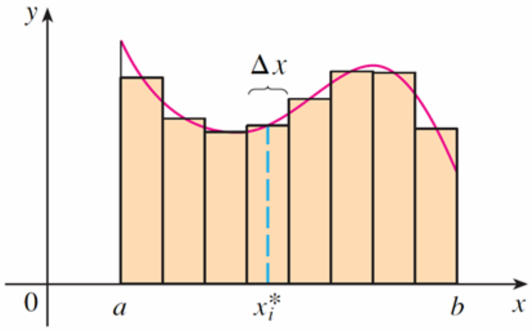


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\Sigma f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

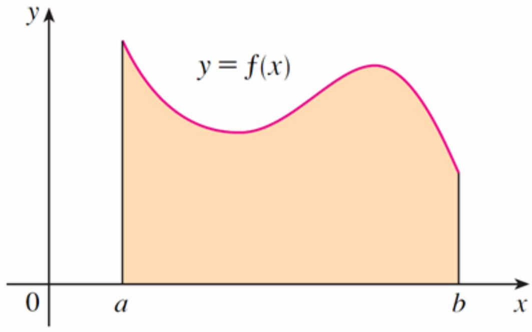


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

What if a function takes on both positive and negative values?

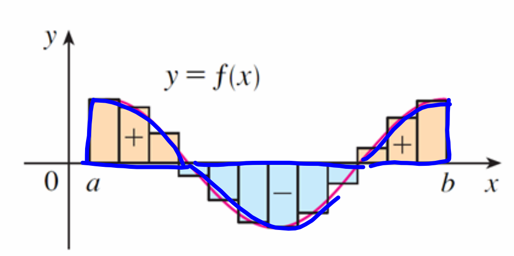


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area

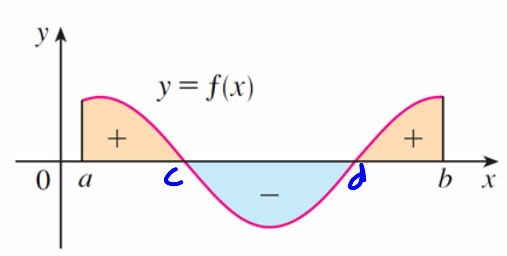


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area

Example: $\int_0^3 \left(x^3 - 6x\right)dx$

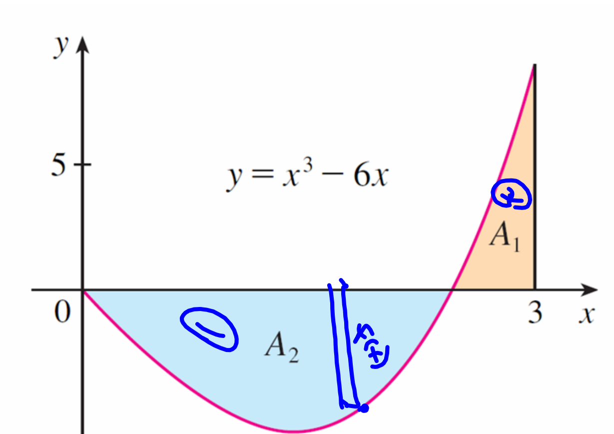


FIGURE 6

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

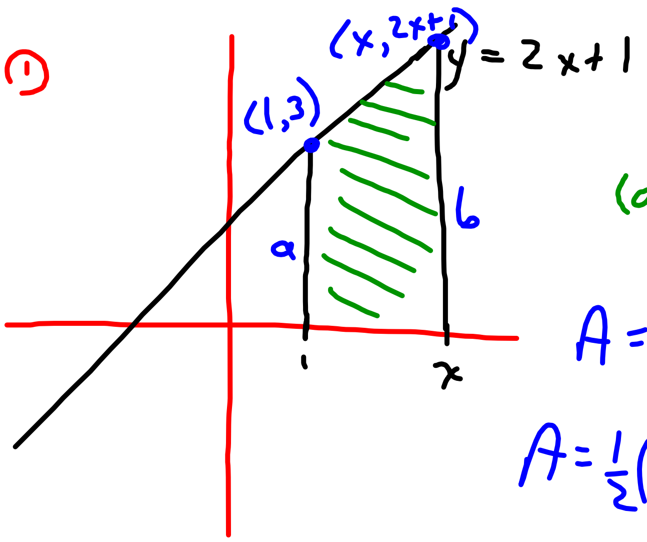
$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\,\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 - 27\left(1 + \frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75$$

1



(a) Find an area function

$A = \frac{1}{2}(a+b)(h)$

$A = \frac{1}{2}(3+2x+1)(x-1)$

$A = \frac{1}{2}(4+2x)(x-1)$

$A = \frac{1}{2}(4x-4+2x^2-2x)$

$A = \frac{1}{2}(2x^2+2x-4)$

$f(x) = 2x+1$

$A = x^2+x-2$

Evaluation Theorem

Evaluation Theorem If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

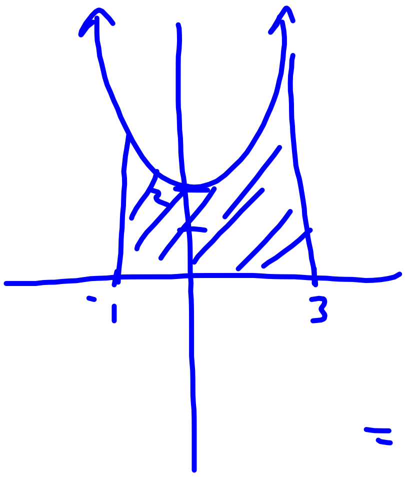
where F is any antiderivative of f, that is, $F' = f$.

$$\int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \tfrac{1}{3}$$

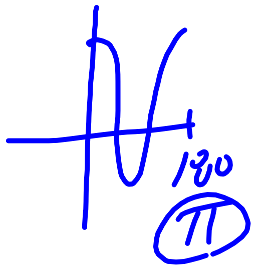
Much easier than using a Riemann sum !!!

Example:

Determine the area below the curve $f(x) = 3x^2 + 2$ between $x = -1$ and $x = 3$.



$$\int_{-1}^{3} (3x^2+2)\,dx$$



$$= x^3 + 2x + C \Big|_{-1}^{3}$$

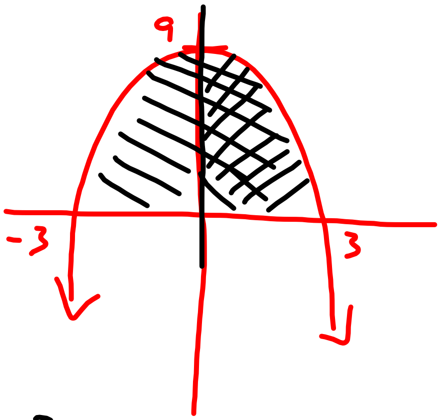
$$= \left[(3)^3 + 2(3) + C\right] - \left[(-1)^3 + 2(-1)\right] + C$$

$$= 33 - (-3)$$

$$\boxed{= 36\,u^2}$$

Example:

Determine the area bound by the curve $y = 9 - x^2$ and the x-axis.



$$\int_{-3}^{3}(9-x^2)\,dx$$

$$= 9x - \frac{x^3}{3}\Big|_{-3}^{3}$$

$$2\int_{0}^{3}(9-x^2)\,dx$$

$$=(27-9)-(-27+9)$$

$$= 18-(-18)$$

$$= \underline{36\,u^2}$$

Indefinite Integrals

- Because of the relation between antiderivatives and integrals, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an *indefinite integral*.
- Thus

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

Example:

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\,\frac{x^5}{5} - 2\tan x + C$$

Indefinite....
No Bounds.!!

$$\int (x^3 - 3x^2 + 4)\,dx$$

$$= \frac{x^4}{4} - x^3 + 4x + C$$

Practice using integrals...

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