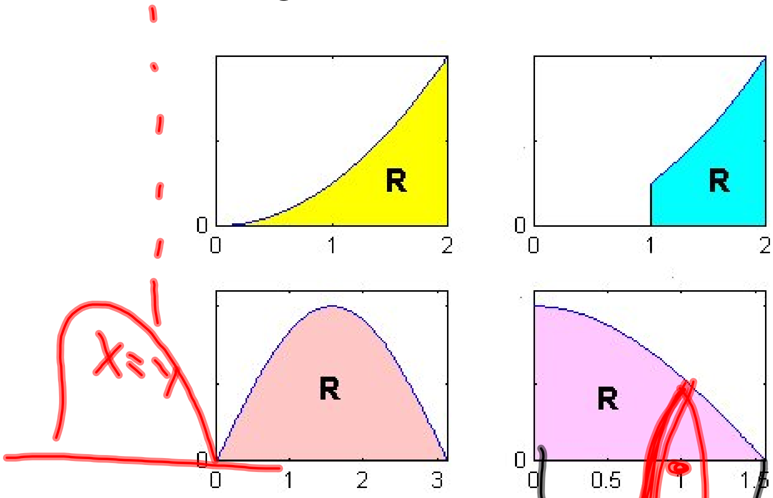
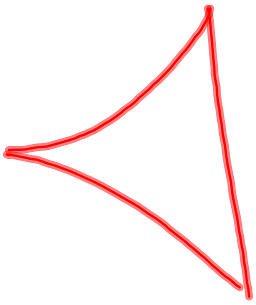
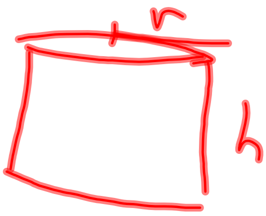
Volumes of Revolution

We will take a region "R" and rotate this region about a vertical line to generate a three dimensional solid.



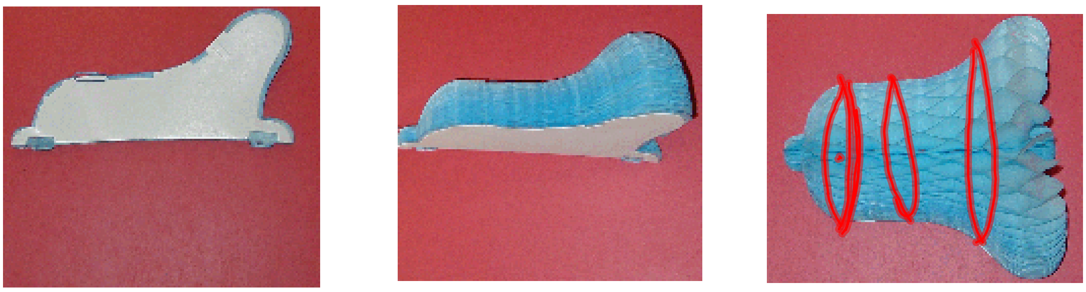




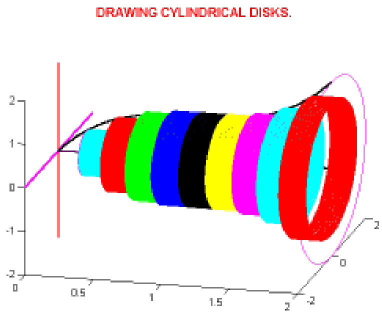
We must be able to visualize these solids of revolution...

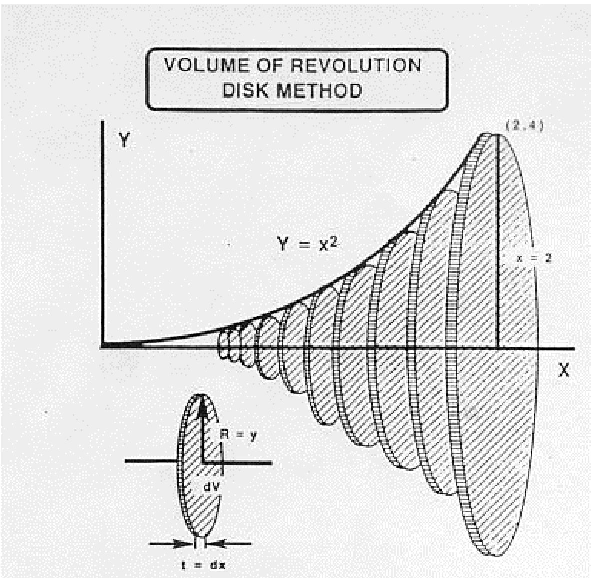
$V=\pi r^2 h$

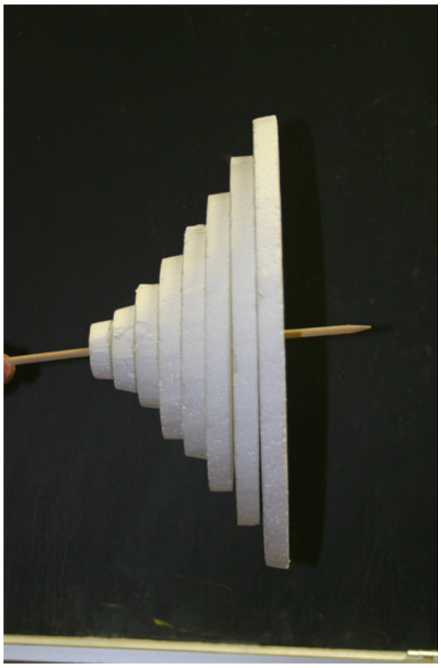
This simple example may help:



Look at the cross-sections of these volumes of revolution...

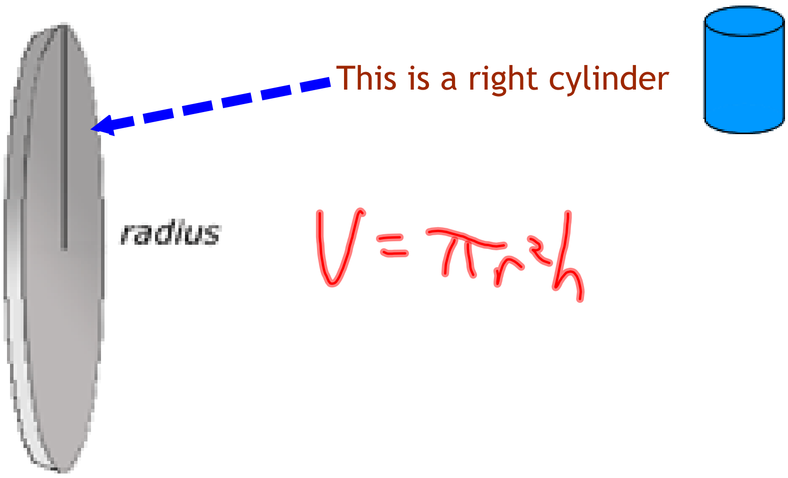






Cross-sections are cylindrical disks

- How do we find the volume of each of these disks?



$V = \pi r^2 h$

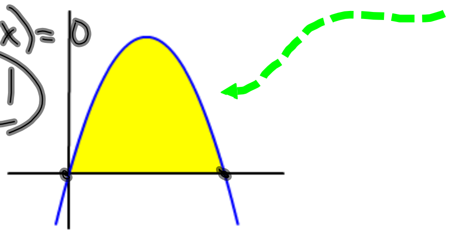
Example:

Determine the volume of the solid generated by rotating the region bound by $y = x - x^2$ and the x-axis, about the x-axis.

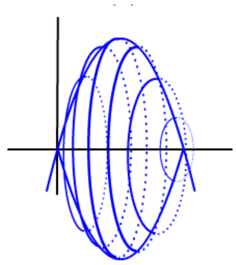
$x - x^2 = 0$

$x(1-x) = 0$

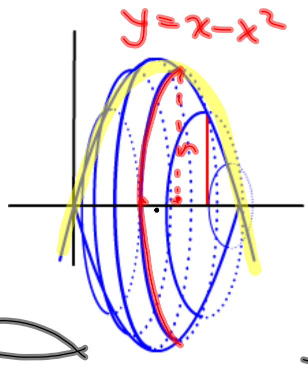
$x = 0, 1$



Here is an animation of this problem



Rotate this graph about the x-axis.



The area of the cross-section is
$A(x) = \pi[f(x)]^2 = \pi(x - x^2)^2$.

The volume of the solid is:

$$\int_0^1 \pi(x - x^2)^2\, dx$$

$$\pi\int_0^1 (x^2 - 2x^3 + x^4)\,dx$$

$$\pi\left(\frac{x^3}{3} - \frac{1}{2}x^4 + \frac{x^5}{5}\right)\Big|_0^1$$

$$= \pi\left[\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) - (0)\right]$$

$$= \left(\frac{10 - 15 + 6}{30}\right)\pi$$

$$= \frac{\pi}{30}\ u^3$$

Washer Method

Example:

Sketch the region bound by the curves $y = x^2$ and $y = x$, and determine the volume of the solid generated by rotating this region about...

(i) the x-axis

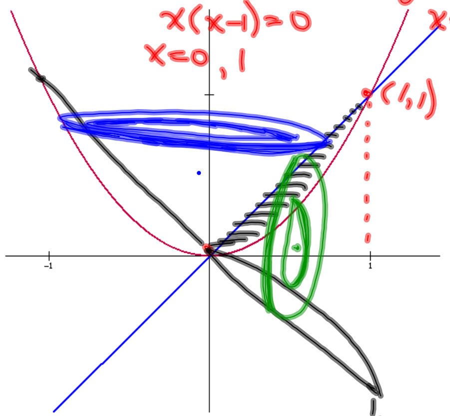
(ii) the y-axis

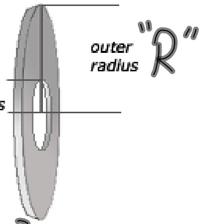
$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$





$\pi R^2 h - \pi r^2 h$

$\pi (R^2 - r^2) h$

$\cancel{(R-r)^2}$

$$V = \pi \int_0^1 \left[(x)^2 - (x^2)^2\right] dx$$

$$V = \pi \int_0^1 (x^2 - x^4)\, dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \left[\left(\frac{1}{3} - \frac{1}{5}\right) - 0\right] \pi$$

$$= \frac{2\pi}{15} u^3$$

(ii) $\pi \int_0^1 (\sqrt{y})^2 - (y)^2\, dy$

$$\pi \int_0^1 (y - y^2)\, dy$$

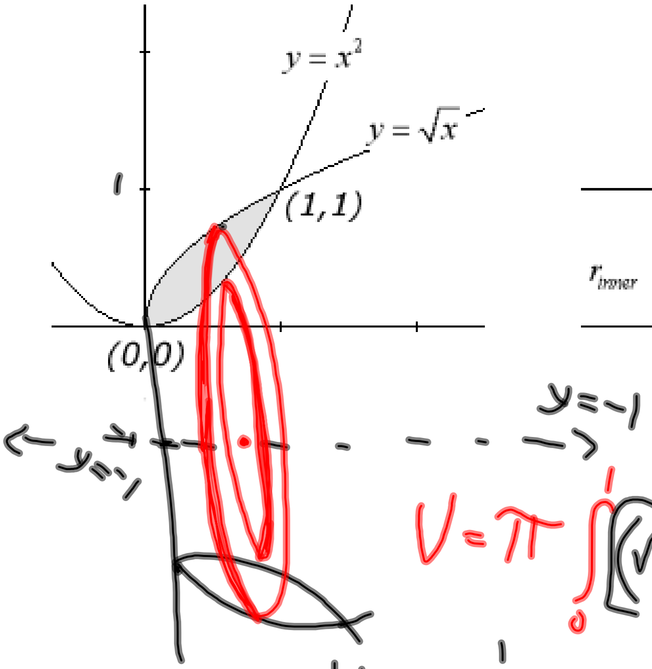
$$\pi \left(y^2/_2 - y^3/_3\right)\Big|_0^1$$

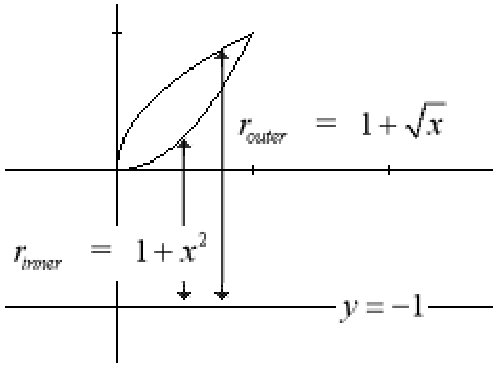
$$= \pi\left(\frac{1}{2} - \frac{1}{3}\right) - (0)$$

$$= \frac{\pi}{6} u^3$$

4) *Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^2$ and $y = \sqrt{x}$ about the line $y = -1$.*

Solution The volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x^2$ about the line $y = -1$ is equal to $\frac{29\pi}{30}$ units cubed.





$$V = \pi \int_0^1 \left[(\sqrt{x}+1)^2 - (x^2+1)^2\right] dx$$

$$V = \pi \int_0^1 (x + 2\sqrt{x} + 1) - (x^4 + 2x^2 + 1)\, dx$$

$$V = \pi \int_0^1 \left(-x^4 - 2x^2 + x + 2\sqrt{x}\right) dx$$

$$V = \pi \left(-\frac{x^5}{5} - \frac{2}{3}x^3 + \frac{x^2}{2} + \frac{4}{3}x^{3/2}\right)\Bigg|_0^1$$

$$\left[\left(-\frac{1}{5} - \frac{2}{3} + \frac{1}{2} + \frac{4}{3}\right) - 0\right]\pi$$

$$\left(\frac{-6-20+15+40}{30}\right)\pi$$

$$= \frac{29\pi}{30} u^3$$

Warm Up

Consider the region below the graph $y = 1 - x^3$ (and above the x-axis) between $x = 0$ and $x = 1$.

(a) Find the area of this region.

(b) Find the volume of the solid "dome" obtained by rotating this region about the y-axis.

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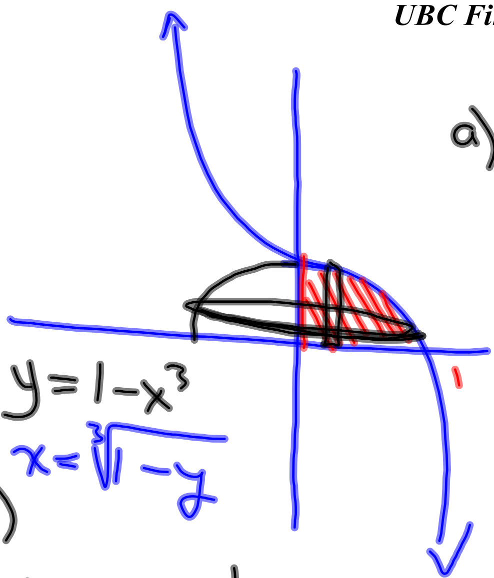

$y = 1 - x^3$

$x = \sqrt[3]{1-y}$

a) $A = \int_0^1 (1 - x^3)\,dx$

$= x - \frac{x^4}{4} \Big|_0^1$

$= 1 - \frac{1}{4} = \underline{\frac{3}{4}u^2}$

b)

$V = \pi \int_0^1 \left(\sqrt[3]{1-y}\right)^2 dy$

$V = -\pi \int_0^1 (1-y)^{2/3}(-1)\,dy$

$= -\pi \left(\frac{3}{5}(1-y)^{5/3}\right)\Big|_0^1$

$= -\pi \left[(0) - \frac{3}{5}(1)\right]$

$\frac{3\pi}{5}u^3$