6. Consider the graph of the function (UNB 2000 Final Exam)

$$f(x) = \frac{(x+2)^2}{x^2+4}.$$

You are given that

$$f'(x) = \frac{16-4x^2}{(x^2+4)^2} \quad \text{and} \quad f''(x) = \frac{8x(x^2-12)}{(x^2+4)^3}.$$

(a) Find the x-intercept.

(b) Find all vertical or horizontal asymptotes, if any.

(c) Determine the intervals where $f(x)$ is increasing or decreasing.

(d) Determine the intervals where $f(x)$ is concave up or concave down.

(e) Find all inflection points.

(f) Find the absolute maximum value of $f(x)$, if it exists.

(g) Sketch the graph of $f(x)$.

$$f(x) = \frac{(x+2)^2}{x^2+4}.$$

$$f'(x) = \frac{16-4x^2}{(x^2+4)^2} \quad \text{and} \quad f''(x) = \frac{8x(x^2-12)}{(x^2+4)^3}$$

Intercepts

x-Int:

$0 = (x+2)^2$

$x = -2$

y-Int

$y = \frac{(0+2)^2}{0^2+4}$

$y = 1$

Asymptotes

Vertical:

$x^2+4 = 0$

$\emptyset$

None

Horizontal:

$$\lim_{x \to \pm\infty} \frac{\frac{x^2}{x^2}+\frac{4x}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}+\frac{4}{x^2}} = \frac{1+0+0}{1+0}$$

$y = 1$

Inc/Dec:

Critical Values of $f'(x)$

$$\frac{16-4x^2}{(x^2+4)^2} = 0$$

$$\frac{4(4-x^2)}{(x^2+4)^2} = 0$$

$$\frac{4(2-x)(2+x)}{(x^2+4)^2} = 0$$

$x = \pm 2$

	$16-4x^2$	$(x^2+4)^2$	f'	f
$(-\infty,-2)$	$-$	$+$	$-$	Dec
$(-2,2)$	$+$	$+$	$+$	Inc
$(2,\infty)$	$-$	$+$	$-$	Dec

Local Max

$(2,2)$

Local Min

$(-2,0)$

Concavity:

$$\frac{8x(x^2-12)}{(x^2+4)^3} = 0$$

$x^2-12=0$

$x^2=12$

$x=\pm\sqrt{12}$

or

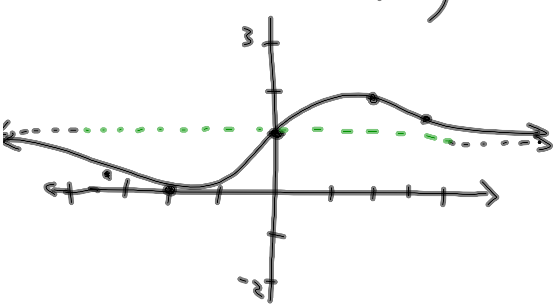
$(x-\sqrt{12})(x+\sqrt{12})=0$

$x = 0, \pm\sqrt{12}$

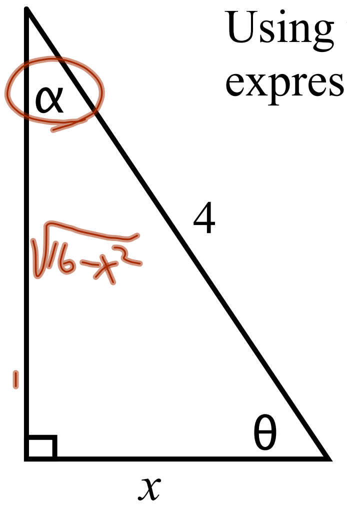
	$8x$	$x-\sqrt{12}$	$x+\sqrt{12}$	f''	f
$(-\infty,-\sqrt{12})$	$-$	$-$	$-$	$-$	Down
$(-\sqrt{12},0)$	$-$	$-$	$+$	$+$	Up
$(0,\sqrt{12})$	$+$	$-$	$+$	$-$	Down
$(\sqrt{12},\infty)$	$+$	$+$	$+$	$+$	Up

Inf. Pts.:

$(-\sqrt{12}, 0.13)$ $(0,1)$ $(\sqrt{12}, 1.8)$



Warm Up



Using the diagram shown determine an expression for each of the following:

$\sin\theta = \frac{\sqrt{16-x^2}}{4}$ $\quad\quad \sec\alpha = \frac{4}{\sqrt{16-x^2}}$

$\tan\alpha = \frac{x}{\sqrt{16-x^2}}$ $\quad\quad \tan\theta = \frac{\sqrt{16-x^2}}{x}$

$x^{-1} = \frac{1}{x}$

arc cos

$\cos^{-1}\left(\frac{x}{4}\right) = \theta$ $\quad\quad \sec^{-1}\left(\frac{4}{\sqrt{16-x^2}}\right) = \alpha$

Not $\frac{1}{\cos}$

$\cos^2 x = (\cos x)^2$

$\cos^{-1} x \neq (\cos x)^{-1}$ $\quad\quad (\cos x)^{-1}$

$\cos^{-1} x \Rightarrow$ Inverse cosine

Derivatives of Transcendental Functions

transcendental functions

(*mathematics*) Functions which cannot be given by any algebraic expression involving only their variables and constants.

Examples include the functions log *x*, sin *x*, cos *x*, e^x and any functions containing them.

Inverse Trigonometric Functions

Let's review the definition of an inverse trigonometric function:

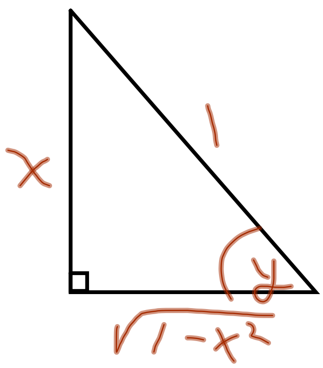
$$y = \sin^{-1} x \qquad \text{or} \qquad y = Arc\sin x$$

What do the above statements mean verbally?

"y is the ~~angle~~ angle whose sine ratio equals x"

Express this visually:

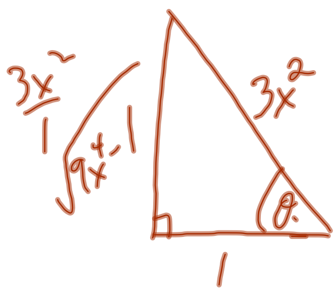
$\cos y = \frac{\sqrt{1-x^2}}{1}$



Example:

- Represent the inverse trigonometric function $\theta = \sec^{-1}(3x^2)$ with a diagram.

- Evaluate: $y = \tan\underbrace{\left[\sec^{-1}(3x^2)\right]}_{\theta}$

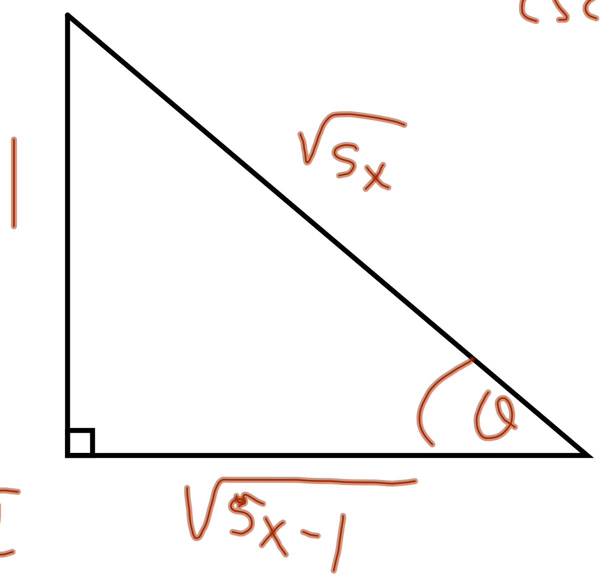


$y = \sqrt{9x^4-1}$

Example:

Evaluate the following: $y = \cos\underbrace{\left[\csc^{-1}\sqrt{5x}\right]}_{\theta}$

$\csc\theta = \frac{hyp}{opp}$



$$= \frac{\sqrt{5x-1}}{\sqrt{5x}}$$

$$= \sqrt{\frac{5x-1}{5x}}$$

$$= \sqrt{1 - \frac{1}{5x}}$$

Differentiating Inverse Trigonometric Functions

Ex. Differentiate $y = \cos^{-1}(2x^2)$

explicit

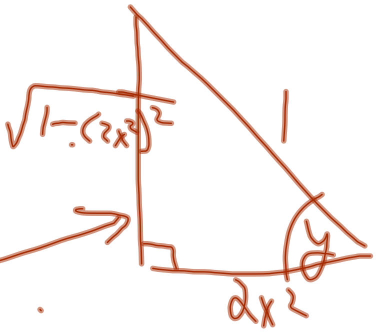
$$\cos y = \cancel{\cos}\left(\cancel{\cos^{-1}}(2x^2)\right)$$

$\tan(\tan^{-1})$

$$\cos y = 2x^2$$

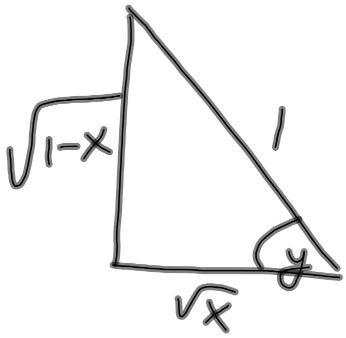
$$-\sin y \frac{dy}{dx} = 4x$$

$$\frac{dy}{dx} = \frac{4x}{-\sin y}$$



$$\frac{dy}{dx} = \frac{4x}{-\sqrt{1-(2x^2)^2}}$$

Ex. Differentiate $y = \cos^{-1}\left(\sqrt{x}\right)$



$$\cos y = \sqrt{x}$$

$$-\sin y\left(\frac{dy}{dx}\right) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-\frac{1}{2}}}{-\sin y}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}}{-\sqrt{1-(\sqrt{x})^2}}$$

$$d(\cos^{-1}u) = \frac{-du}{\sqrt{1-u^2}}$$

These two examples lead to the following set of rules for differentiating inverse trigonometric functions:

$$\frac{d\left(\sin^{-1} u\right)}{du} = \frac{1}{\sqrt{1-u^2}}\,du \qquad \frac{d\left(\csc^{-1} u\right)}{du} = \frac{-1}{u\sqrt{u^2-1}}\,du$$

$$\frac{d\left(\cos^{-1} u\right)}{du} = \frac{-1}{\sqrt{1-u^2}}\,du \qquad \frac{d\left(\sec^{-1} u\right)}{du} = \frac{1}{u\sqrt{u^2-1}}\,du$$

$$\frac{d\left(\tan^{-1} u\right)}{du} = \frac{1}{1+u^2}\,du \qquad \frac{d\left(\cot^{-1} u\right)}{du} = \frac{-1}{u^2+1}\,du$$

Examples:

Differentiate each of the following...

$$f(x) = x^3 \sin^{-1}\left(3x^2\right)$$

$$f(x) = \sqrt{3x - \tan^{-1}\sqrt{x}}$$