These two examples lead to the following set of rules for differentiating inverse trigonometric functions:

$$\frac{d\left(\sin^{-1}u\right)}{du}=\frac{1}{\sqrt{1-u^2}}du \qquad \frac{d\left(\csc^{-1}u\right)}{du}=\frac{-1}{u\sqrt{u^2-1}}du$$

$$\frac{d\left(\cos^{-1}u\right)}{du}=\frac{-1}{\sqrt{1-u^2}}du \qquad \frac{d\left(\sec^{-1}u\right)}{du}=\frac{1}{u\sqrt{u^2-1}}du$$

$$\frac{d\left(\tan^{-1}u\right)}{du}=\frac{1}{1+u^2}du \qquad \frac{d\left(\cot^{-1}u\right)}{du}=\frac{-1}{u^2+1}du$$

Examples:

Differentiate each of the following...

$$f(x)=x^3\sin^{-1}\left(\overset{\text{"u"}}{3x^2}\right)$$

$$f'(x)=3x^2\sin^{-1}(3x^2)+x^3\left(\frac{6x}{\sqrt{1-9x^4}}\right)$$

$$f(x)=\sqrt{3x-\tan^{-1}\sqrt{x}}$$

$$f'(x)=\frac{1}{2}\left[3x-\tan^{-1}\sqrt{x}\right]^{-\frac{1}{2}}\left[3-\frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right]$$

$$\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+x}$$

$$f(x)=\frac{\cot^3 5x}{\cot^{-1}(5x)} \rightarrow (\cot 5x)^3$$

$$f'(x)=\frac{\left[3(\cot 5x)^2(-\csc^2 5x)(5)\right]\cot^{-1}5x - (\cot^3 5x)\left[\frac{-5}{1+25x^2}\right]}{\left[\cot^{-1}(5x)\right]^2}$$

$$f(x)=\tan\left[\underset{\csc^{-1}}{\underline{arc}}\csc\left(x^5\right)\right]$$

$$f'(x)=\sec^2\left[\csc^{-1}(x^5)\right]\left[\frac{-5x^4}{x^5\sqrt{x^{10}-1}}\right]$$

Homework:

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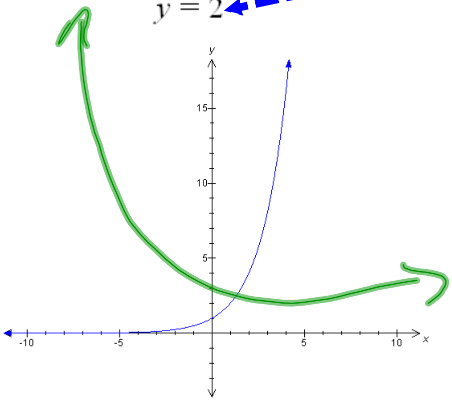
#1, 2, 4, 6

Differentiating Exponential Functions

What is an exponential function?

$y = 2^x$ Base must be a Real number



When you do not have a rule to differentiate resort to the definition...

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = a^x$
$f(x+h) = a^{x+h}$

Let's try and differentiate $y = a^x$, $a \in R$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

This factor does not depend on *h*, therefore we can move to the front of the limit

Thus we now have...

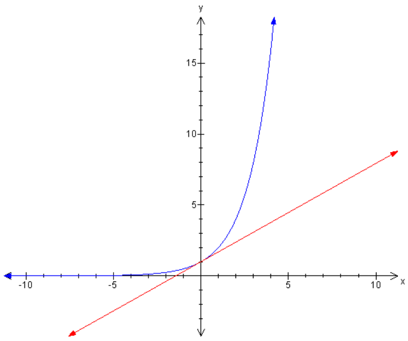
$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

What would be the value of $f(0)$?

$a^0 = 1$

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

What would this represent in terms of slope??



$m = f'(0)$

We have determined that $f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$

Same thing!

and that $\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "*e*"...Euler's Number

```
e^(1)
        2.718281828
```

This leads to the following definition...

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$y = e^x$

What does this mean geometrically?

- Geometrically, this means that
  - of all the exponential functions $y = a^x$,
  - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 .

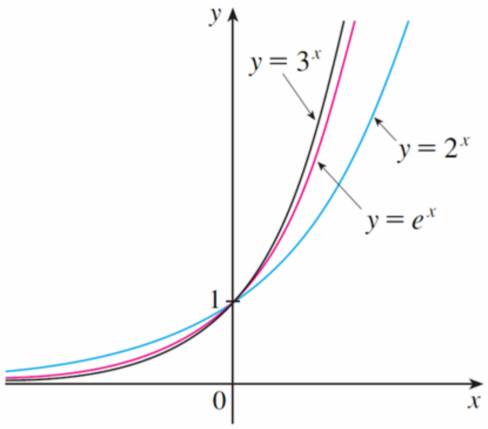


FIGURE 6

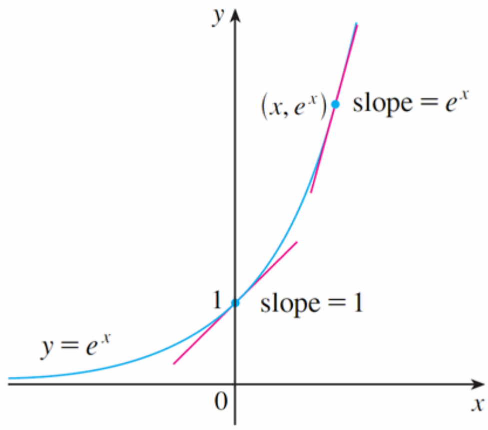


FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x \quad (1$$

slope of tangent

This is the ONLY function that is its own derivative

$$f(x) = e^x$$

$$f'(x) = e^x$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$