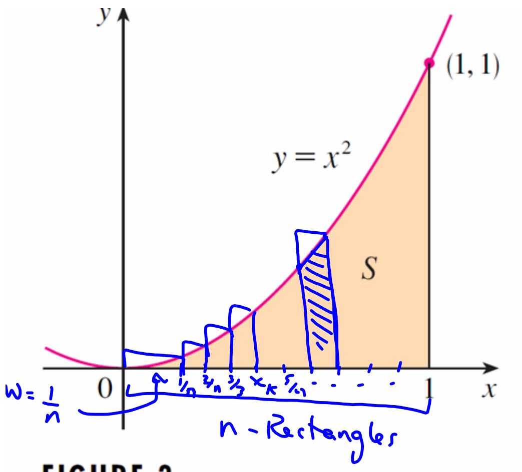


FIGURE 3

We want to find the area below this curve using "n" rectangles.

What will be the width of each rectangle? $\Delta x = \frac{1-0}{n} = \frac{1}{n}$

How will we determine the height of each rectangle? x_k

$x_k = 0 + (\Delta x)k$

$\boxed{x_k = \frac{k}{n}}$

of the Rectangle $\to f(x_k)$

Write out an expression for the area of these "n" rectangles?

$$A = \sum_{k=1}^{n} \left(\frac{1}{n}\right) f(x_k) = \frac{1}{n}f(x_1) + \frac{1}{n}f(x_2) + \frac{1}{n}f(x_3) + \dots$$

$$A = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \quad ?? \quad \boxed{f(x) = x^2}$$

$$f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^2$$

$$A = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2$$

$$A = \frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{n^2} = \frac{1}{n^2}(k^2)$$

$$A = \frac{1}{n^3}\boxed{\sum_{k=1}^{n} k^2} = 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2$$

$$\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$A = \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$A = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$ ← What does "n" Represent??

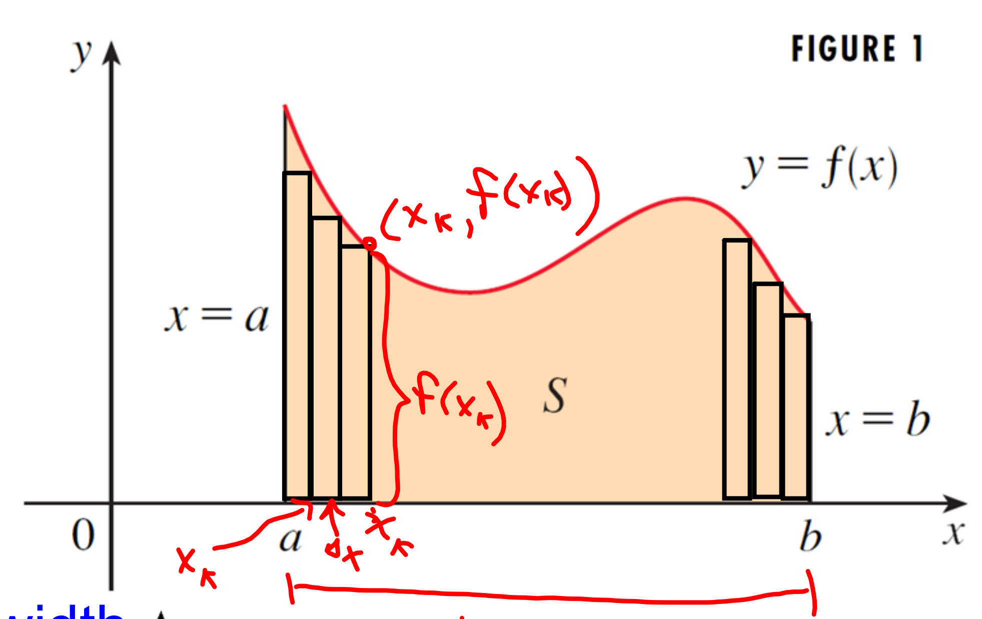
$$\lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)$$

$$= \frac{1}{3} + 0 + 0$$

$$\boxed{= \frac{1}{3}u^2}$$

Riemann Summation

Develop a general formula for the area below a curved surface using "n" rectangles.



width Δx

$$\boxed{\Delta x = \frac{b-a}{n}}$$

height x_k

$x_k = a + (\Delta x)k$

$x_k = a + \left(\frac{b-a}{n}\right)k$

area

$$A = \Delta x \sum_{k=1}^{n} f(x_k)$$

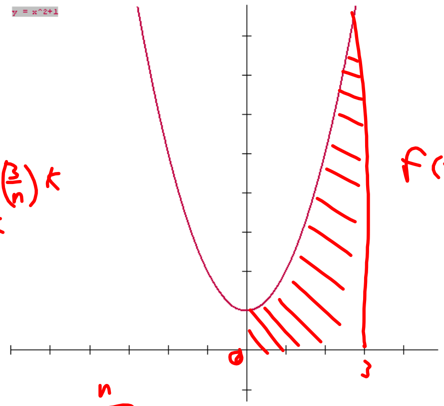
$$\Delta x = \frac{b-a}{n} \qquad x_k = a + (\Delta x)k$$

Use a Riemann Summation to determine the area below the curve $y = x^2 + 1$, between $x = 0$ and $x = 3$.

$\Delta x = \frac{3}{n}$

$x_k = 0 + \left(\frac{3}{n}\right)k$

$x_k = \frac{3k}{n}$



$f(x) = x^2 + 1$

$$A = \frac{3}{n}\sum_{k=1}^{n} f\left(\frac{3k}{n}\right)$$

$$A = \frac{3}{n}\sum_{k=1}^{n}\left[\left(\frac{3k}{n}\right)^2 + 1\right]$$

$$A = \frac{3}{n}\sum_{k=1}^{n}\left(\frac{9k^2}{n^2} + 1\right)$$

$\sum(a+b) = \sum a + \sum b$

$$A = \frac{3}{n}\left[\sum_{k=1}^{n}\frac{9k^2}{n^2} + \sum_{k=1}^{n} 1\right]$$

$\sum_{k=1}^{n} c = \underline{\underline{cn}}$

$$A = \frac{3}{n}\left[\frac{9}{n^2}\boxed{\sum_{k=1}^{n} k^2} + 1n\right]$$

$\sum_{k=1}^{4} 3 = 3+3+3+3$

$= 3 \times 4 = 12$

$$A = \frac{3}{n}\left[\frac{9}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + 1n\right]$$

$$A = \frac{3}{n}\left(3n + \frac{9}{2} + \frac{3}{2n} + n\right)$$

$$A = 9 + \frac{27}{2n} + \frac{9}{2n^2} + 3$$

$$A = 12 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\lim_{n\to\infty}\left(12 + \frac{27}{2n} + \frac{9}{2n^2}\right)$$

$\boxed{A = 12u^2}$

$$\int_0^3 (x^2+1)\,dx$$

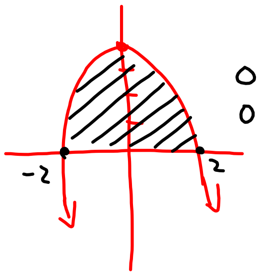
$$= \frac{x^3}{3} + x \Big|_0^3$$

$$= (9+3) - (0)$$

$$= 12u^2$$

Use a Riemann sum to determine the area bound by the curve $f(x) = -x^2 + 4$ and the x-axis.

"closed in by"



$0 = 4 - x^2$

$0 = (2-x)(2+x)$

$x = \pm 2$

$\Delta x = \frac{4}{n} \qquad x_k = -2 + \frac{4k}{n}$

$$A = \frac{4}{n}\sum_{k=1}^{n} f\left(-2+\frac{4k}{n}\right) \qquad f(x) = -x^2+4$$

$$A = \frac{4}{n}\sum_{k=1}^{n}\left[-\left(-2+\frac{4k}{n}\right)^2 + 4\right]$$

$$A = \frac{4}{n}\sum_{k=1}^{n} -\left(4 - \frac{16k}{n} + \frac{16k^2}{n^2}\right) + 4$$

$$A = \frac{4}{n}\sum_{k=1}^{n}\left(\cancel{-4} + \frac{16k}{n} - \frac{16k^2}{n^2} \cancel{+4}\right)$$

$$A = \frac{4}{n}\left[\sum_{k=1}^{n}\frac{16k}{n} - \sum_{k=1}^{n}\frac{16k^2}{n^2}\right]$$

$$A = \frac{4}{n}\left[\frac{16}{n}\sum_{k=1}^{n} k - \frac{16}{n^2}\sum_{k=1}^{n} k^2\right)$$

$$A = \frac{4}{n}\left[\frac{16}{n}\left(\frac{n^2}{2}+\frac{n}{2}\right) - \frac{16}{n^2}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\right)$$

$$A = \frac{4}{n}\left(8n \cancel{+8} - \frac{16n}{3} \cancel{-8} - \frac{8}{3n}\right)$$

$$A = \frac{32}{1} - \frac{64}{3} - \frac{32}{3n^2}$$

$$\lim_{n\to\infty}\left(\frac{32}{3} - \frac{32}{3n^2}\right)$$

$$\boxed{A = \frac{32}{3}u^2}$$

Example:

Suppose ***R*** is the area in the first quadrant below the curve
$y = 2x^2 + 2$ between $x = 1$ and $x = 5$.

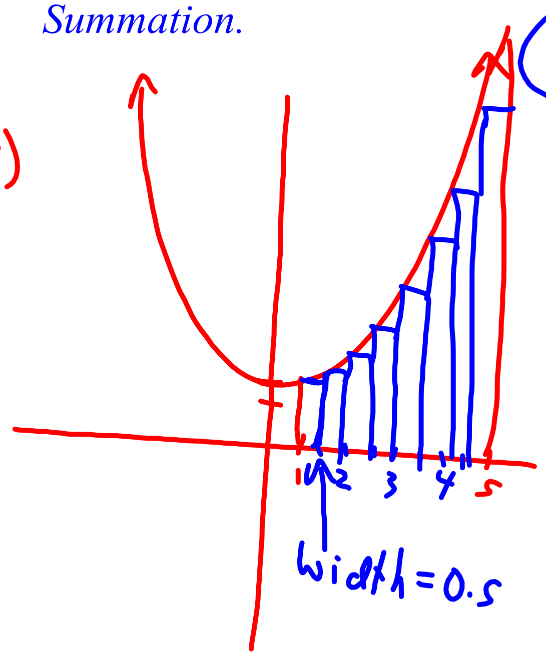
(a) Approximate the area of "R" using 8 rectangles with height determined by the left-hand endpoint

(b) Determine the exact area of "R" using a Riemann Summation.

$A = 90\frac{2}{3}$

$\boxed{A = \frac{272}{3} u^2}$

(a)



$y = 2x^2 + 2$

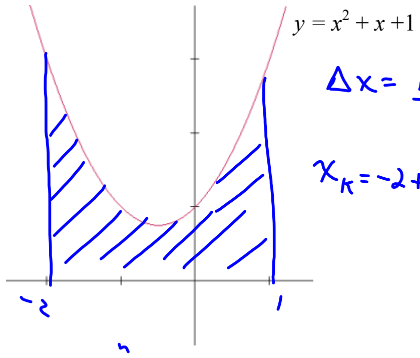
x	y
1	4
1.5	6.5
2	10
2.5	14.5
3	20
3.5	26.5
4	34
4.5	42.5

$A = 0.5(4 + 6.5 + 10 + 14.5 + 20 + 26.5 + 34 + 42.5)$

$A = \underline{\underline{79 u^2}}$

Check-up...

Use a Riemann Summation to determine the area below the curve $y = x^2 + x + 1$ and above the x-axis, between x = −2 and x = 1.



$$\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}$$

$$x_k = -2 + \frac{3}{n}k$$

$$A = \Delta x \sum_{k=1}^{n} f(x_k)$$

$$A = \frac{3}{n} \sum_{k=1}^{n} f\left(-2 + \frac{3k}{n}\right) \longleftarrow f(x) = x^2 + x + 1$$

$$A = \frac{3}{n} \sum_{k=1}^{n} \left[\left(-2 + \frac{3k}{n}\right)^2 + \left(-2 + \frac{3k}{n}\right) + 1\right]$$

$$A = \frac{3}{n} \sum_{k=1}^{n} \left(4 - \frac{12k}{n} + \frac{9k^2}{n^2} - 2 + \frac{3k}{n} + 1\right)$$

$$A = \frac{3}{n} \sum_{k=1}^{n} \left(\frac{9k^2}{n^2} - \frac{9k}{n} + 3\right)$$

$$A = \frac{3}{n}\left[\sum_{k=1}^{n} \frac{9k^2}{n^2} - \sum_{k=1}^{n} \frac{9k}{n} + \sum_{k=1}^{n} 3\right]$$

$$A = \frac{3}{n}\left[\frac{9}{n^2}\sum_{k=1}^{n} k^2 - \frac{9}{n}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 3\right]$$

$$\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$A = \frac{3}{n}\left[\frac{9}{n^2}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) - \frac{9}{n}\left(\frac{n^2}{2} + \frac{n}{2}\right) + 3n\right]$$

$$A = \frac{3}{n}\left[3n + \cancel{\frac{9}{2}} + \frac{3}{2n} - \frac{9n}{2} - \cancel{\frac{9}{2}} + 3n\right]$$

$$A = \frac{3}{n}\left(6n + \frac{3}{2n} - \frac{9n}{2}\right)$$

$$A = 18 + \frac{9}{2n^2} - \frac{27}{2} \longleftarrow$$

"n" Represents the # of rectangles to be stacked within shaded area.

$$\lim_{n \to \infty} \frac{18}{1} - \frac{27}{2} + \frac{9}{2n^2} \to 0$$

$$= \frac{36}{2} - \frac{27}{2}$$

$$= \frac{9}{2} u^2$$

Practice Exercises:

Determine each of the following areas using a Riemann Sum with an infinite number of rectangles:

(1) Area below $f(x) = 2x^2 - 1$ between $x = 1$ and $x = 3$.

(2) Area below $f(x) = x^2 - 4x + 7$ between $x = -1$ and $x = 2$.

(3) Area below $f(x) = x^3 + 2x^2 + x$ between $x = 0$ and $x = 1$.

The Distance Problem

Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30 second time interval. We take speedometer readings every 5 seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28

Here is the data in *ft*./s...

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	46	41

$(25 \times 5) + (31 \times 5) + (35 \times 5) + (43 \times 5) + (47 \times 5) + (46 \times 5) = 1135 \text{ ft}$

Remember the formula to calculate distance...

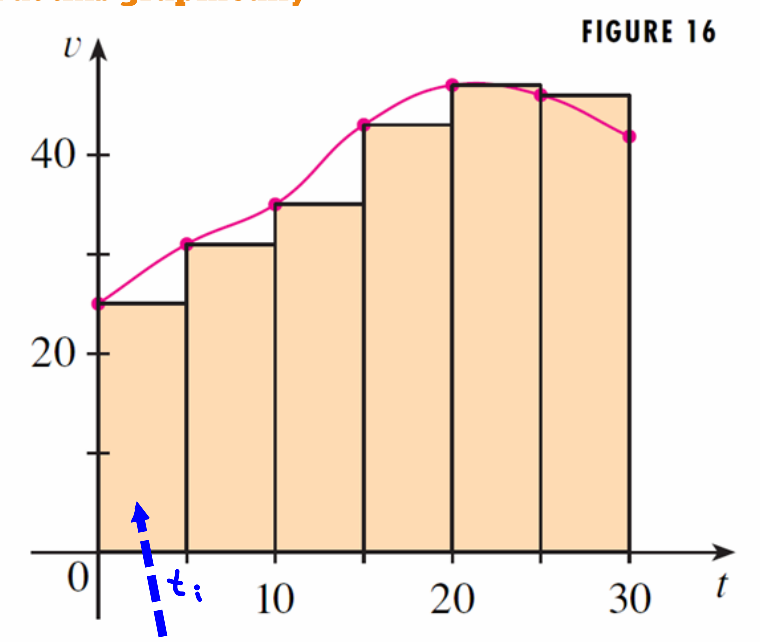
$d = velocity \times time$

We now want to estimate the distance travelled during each 5 second interval...

- Let's use the starting time of each interval as our estimate of the velocity...

Does this process remind you of anything we have done recently???

Look at this graphically...



Look at the area of each of these rectangles...what do you notice??

In general, using left endpoints the area of each rectangle would be...

$$f(t_0)\,\Delta t + f(t_1)\,\Delta t + \cdots + f(t_{n-1})\,\Delta t = \sum_{i=1}^{n} f(t_{i-1})\,\Delta t$$

Using right endpoints the area of each rectangle would be...

$$f(t_1)\,\Delta t + f(t_2)\,\Delta t + \cdots + f(t_n)\,\Delta t = \sum_{i=1}^{n} f(t_i)\,\Delta t$$

The more frequently we measure the velocity, the more accurate our estimates become.

The exact distance traveled would be...

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\,\Delta t = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t$$

IMPORTANT CONNECTION....

The area below a velocity-time graph between two particular times would represents the distance traveled between those times.

Refresher...

Determine the general antiderivative:

$5x^4$ $\sqrt[5]{x^7}$

$x^3 e$

$x^{1/2}$ $\frac{1}{2}x^{-\frac{1}{2}}$

$$f'(x)=\frac{5x}{(7x^2-2)^4}-\frac{\csc\sqrt{x}\cot\sqrt{x}}{\sqrt{x}}+\frac{\sin 2x}{\sqrt{1-\cos^2(2x)}}-x^3e^{5x^4}+\frac{3x-2}{3x^2-4x+1}+\sqrt[5]{x^7}-8\pi$$

$$f'(x)=\frac{5}{14}(7x^2-2)^{-4}\,14x-\csc\sqrt{x}\cot\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)^{-\frac{1}{2}}+\frac{-\sin(2x)(2)}{\sqrt{1-(\cos 2x)^2}}$$

$$+\frac{1}{2}\left(\frac{6x-4}{3x^2-4x+1}\right)-\frac{1}{20}e^{5x^4}(20x^3)+x^{7/5}-8\pi$$

$$f(x)=-\frac{5}{42}(7x^2-2)^{-3}$$